resistive circuit theory

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To

Kathleen

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PREFACE

Another freshman level book on circuit theory? Yes. Is it significantly different from all the others? Yes. Two departures distinguish the present text: the topics treated are restricted to resistive circuits, and the mathematical notation used is restricted to a simple well-defined system, APL.

APL is a simplification and extension of the familiar notation of vector algebra which extends its domain of application far beyond linear algebra, and enhances the role of arrays (vectors, matrices, etc.) as an important tool of organized thought. It is also a computer language, and every mathematical expression occurring in the text can be entered without change and executed on a typewriter-like APL terminal.

Three characteristics of the APL notation merit mention here:

1. It has the ability, as a notation, to suggest useful generalizations of functions and relations already defined, and thereby entend their range of validity or shed further light on their underlying structure. It is exciting to experience this aspect of APL, and one is reminded of Bertrand Russell's saying that "A good notation has a subtlety and suggestiveness which at times makes it seem almost like a live teacher".

2. The executability of APL permits a student at an APL terminal to readily experiment with a wide variety of interesting circuits, and thereby gain insight into circuit behaviour and the structure of circuit theory. Theorems can be discovered rather than encountered first via statement and proof. Speculative ideas can easily be tested.

3. Because of its simple syntax and similarity to vector algebra, APL is easy to learn; one hour spent exploring the language at an interactive terminal is sufficient to give the student or teacher a feel for the language and enable him to start using it. The importance of this ease of learning will be apparent to any teacher who has examined computer-oriented texts on circuit theory which require the student to use two systems of notation (one for the computer and one for the blackboard) and to learn inessential details of the mechanics of computers.

For the beginner - either teacher or student - who seeks a quick initial appreciation of APL, Appendix 1 contains an annotated sample terminal session. Perusal of this session or, better still, its repetition at an APL terminal, should provide the beginner with a feel for the

language and an acquaintance with many of the primitive functions encountered in this book. Appendix 2 is not intended for the beginner; it is a concise summary of all the APL primitive and mixed functions and provides reference material for the reader who has begun to make serious use of APL. What form should an organized introduction to APL for students take? My own recommendation is an approach in which terminal experience constitutes the main activity, with formal lecturing kept to a minimum. A successfully tested example of this approach is offered by Kenneth Iverson's "An Introduction to APL for Scientists and Engineers", a short note* whose theme is one of guided exploration.

One significant aspect of the text which derives from the use of APL is its algorithmic and experimental flavour. At each stage in his understanding of the theory, the student is able to obtain experience of the pertinent relationships, both by executing the expressions appropriate to specific circuit examples, and by incorporating the expressions in defined functions which are then available as building blocks in further analyses. Moreover, the function definitions serve to summarize, in a precise and readable symbolic form, important relations which the student has just learned and which he will later wish to use as tools. Suitably graded examples and exercises serve to develop the student's ability to compose programs, or function definitions, as they are called in APL. I would not be surprised if such an algorithmic treatment of circuit theory led to a more ready acceptance of the subject by the more practically inclined student.

When planning this text I realized that many of the central theorems and concepts of circuit theory might best be introduced and explored within the simpler context of purely resistive circuits, leaving the dynamic behaviour of circuits incorporating inductive and capacitive elements to be treated in a subsequent (or companion) course. The inclusion of both nonreciprocal and nonlinear resistive circuits gives the student a broad and realistic experience of circuit theory. Indeed, in subject matter and approach the present text resembles the first of two courses (Resistive Circuits, and Dynamic Circuits) proposed by the COSINE Committee Commission on Engineering Education (September, 1968).

Although the material of the syllabus does not depart very much in outline from what is conventional, it differs significantly in detail. This first becomes obvious in Chapter 2, where two-terminal components are modelled by APL functions, thereby allowing the student to systematically and easily explore the nature of a component by examining

* Iverson, K. E., "An Introduction to APL for Scientists and Engineers", IBM Philadelphia Scientific Center, Technical Report No. 320-3019, March, 1973.

the response pattern produced by a regular ordering of the excitation. He is also encouraged to build his own models, based on his measurements, and to test these models against the original functions which represented the components. Discussion is not restricted to linear resistances - exponential diodes and polynomial nonlinearities are also explored. The idea of the inverse function (yielding the voltage response to current excitation) is also introduced at this point, in preparation for its extension to the three-terminal component in Chapter 6 and its use in formal circuit analysis in Chapter 8.

Chapter 3 considers the constraints imposed by the interconnection of components. Sources and power are discussed in Chapter 4, and signals in Chapter 5. Chapter 6 concerns the measurement and modelling of 3-terminal components, introduces the use of matrices in the description of the voltage-current relation of a component, and extends the notion to n-terminal components. The idea of regarding an n-terminal circuit as an n-terminal component lays the groundwork for the later treatment of formal circuit analysis. This chapter, like Chapter 2, also gives extensive consideration to nonlinear components, in particular the transistor described by an Ebers-Moll model. Chapter 7 employs arrays in the description of circuit topology and in the concise expression of Kirchhoff's laws in terms of the incidence matrix.

Up to this point, both linear and nonlinear components have been considered in each chapter. Chapters 8, 9 and 10, however, concentrate on circuits composed solely of linear homogeneous components, show how the circuit equations are formulated and solved, and how the solution is interpreted in terms of circuit properties. Here again, the use of arrays is fully exploited, as is the common basis of component and circuit description.

Chapter 11 considers the analysis of a nonlinear resistive circuit, and describes the Newton-Raphson iterative method of analysis. By drawing upon concepts introduced earlier in the text it has been found possible to keep discussion of this topic at an acceptably simple level. A particularly useful consequence of the use of APL is the fact that the Newton-Raphson algorithm, embodied in a brief, readable function, can be introduced very simply in the context of a 2-terminal nonlinear component and then, with minimal change, be generalized to handle the analysis of an arbitrary nonlinear circuit.

Chapter 12 brings together the nonlinear circuit analysis and the analysis of linear homogeneous circuits in the treatment of the small-signal behaviour of nonlinear circuits. This chapter serves an additional purpose in calling upon the student's knowledge of the variety of topics encountered in earlier chapters.

The APL notation was invented by Kenneth Iverson, now of IBM Corporation, while teaching at Harvard, and an early description appears in his book A Programming Language (hence the name APL). Later work with Adin Falkoff at IBM led to refinement and extension of the notation and, in 1966, to its implementation as a language for conversational computing, since when its popularity and use has rapidly increased. It was Kenneth Iverson who, as an IBM Fellow, invited me to work with him at the IBM Philadelphia Scientific Center in order to explore the potential of APL for the teaching of circuit theory. I found this to be a stimulating experience. By his infectious enthusiasm, his quick grasp of circuit theory and, most of all, his appreciation of what teaching is all about, Iverson contributed immensely to the development of this text; it is a pleasure to record my gratitude to him.

Many other people have contributed to this text. John Goacher of Imperial College helped in the formulation of Chapter 11. Other permanent and guest workers at the Philadelphia Scientific Center were a constant source of encouragement, and I particularly wish to acknowledge the contribution of Professor Donald McIntyre of Pomona College in this respect. I would also like to thank Elizabeth Llanso for her excellent handling of the manuscript under somewhat harassing conditions, and Mr. E. E. McDonnell for a critical reading of the manuscript. Finally, I wish to express my gratitude to Mr. Clenyg Squire of IBM United Kingdom Ltd. for his cooperation in this endeavour.

In its restriction to resistive circuits, its use of APL notation, and its implicit assumption of the value of mathematical experimentation using APL terminals, the present text must be regarded as experimental. Indeed, my own experimental use of the text is at present under way, and I look forward to receiving constructive criticism from both students and colleagues.

Robert Spence

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CHAPTER 1

CIRCUIT DESIGN

Anyone who is encouraged to study circuit theory deserves a succinct preliminary discussion of the value of the subject and the context within which it is relevant. Only then will the associated ideas and methods be seen in their proper perspective. The present chapter attempts this and will often be referred to later to supply the motivation or context for a particular discussion.

In its normal engineering context, circuit theory is viewed as both an analytical and conceptual tool of considerable value in the design of electric circuits. It is also of wider interest as a tool for the study of some non-electrical circuits. It is, however, in the context of electrical circuit design that the subject of circuit theory will be presented in this book, and the needs of the circuit designer will provide the underlying motivation.

The designer of any object - whether it be an electrical circuit or an automobile - is supplied with a specification of the properties to be exhibited by that object, and must construct it by interconnecting a number of building blocks. The object is the *product* of the designer, and the building blocks are his *ingredients*.

Specifically, a *circuit designer* is a person who has access to a range of component types (e.g., resistors, transistors, wires) and who connects these components together in such a way as to make a *circuit* - such as an amplifier - which exhibits useful behaviour. In turn, a *system designer* will take these and other circuits and connect them together to make a larger circuit usually referred to as a *system*. To take a familiar example, components such as resistors and transistors may be suitably interconnected to form amplifiers and switches, which may then be connected to other circuits (e.g., telephone sets) to form a telephone system. Thus, in one respect, the telephone system is merely a collection of components. What will be seen to be important, however, is the idea of regarding a collection of interconnected components as a *new component* whose interior is of little interest but whose external characteristics are important.

This idea is reinforced by considering the functions of the circuit designer and system designer in the reverse order to that just discussed. In practice, the system designer specifies the function that he wishes a circuit to perform. The general form of this specification is shown in Fig. 1.1; the 4-terminal box is required to exhibit a specified voltage V between the pair of terminals X and Y when a current of value I flows between terminals A and B. Such is the system designer's view of the circuit he requires. The circuit designer's view of the same circuit is illustrated in Fig. 1.2; he is concerned with choosing components and their interconnection in such a way that when the current I flows the voltage V is generated. Normally there will be many different ways of choosing the components and interconnecting them to satisfy the system designer's specification.

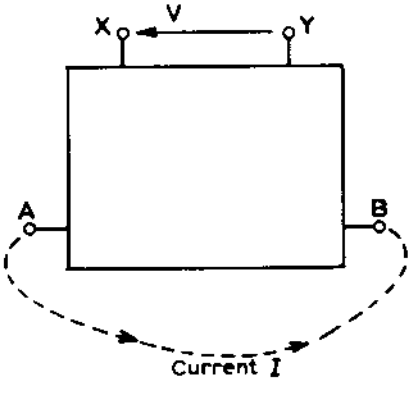


Fig. 1·1

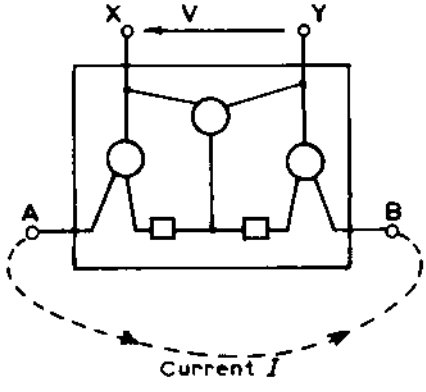


Fig. 1·2

The illustration of Fig. 1.3 shows the relative positions and functions of the circuit and system designer, and of the designer of the components used as ingredients by the circuit designer. At each interface the product of one designer constitutes some or all of the ingredients available to the other. Since the components manufactured by the component designer are identical with those contained within the circuits used as building blocks by the system designer, it could be remarked that each designer is concerned with much the same material. This is essentially true; what distinguishes the three categories of designer shown in Fig. 1.3 is their view of the product they are designing. The system designer, for example, is interested principally in the external properties (e.g., power gain, bandwidth) of an amplifier, whereas the circuit designer must concern himself with the voltages and currents associated with its internal components. He, in turn, is unconcerned with those underlying physical, metallurgical and chemical principles which lie in the domain of interest of the component designer.

The distinction between each of the three levels shown in Fig. 1.3 is principally defined by the quantities of interest to each designer. At the level of component design, quantities such as majority and minority carrier concentrations, diffusion rates and impurity profiles have been found to be useful descriptors in the processes and products involved. At the other extreme, the interest of the system designer in circuit properties such as power gain and bandwidth has already been noted. Between these two levels, the circuit designer finds it most helpful to be concerned with the voltage and current associated with each component, both in isolation and when interconnected with others to form a circuit. It is in this context that the discipline known as <u>circuit theory</u> is relevant, for it encompasses the principles and techniques which allow us to describe and predict the voltage-current behaviour of an electrical circuit.

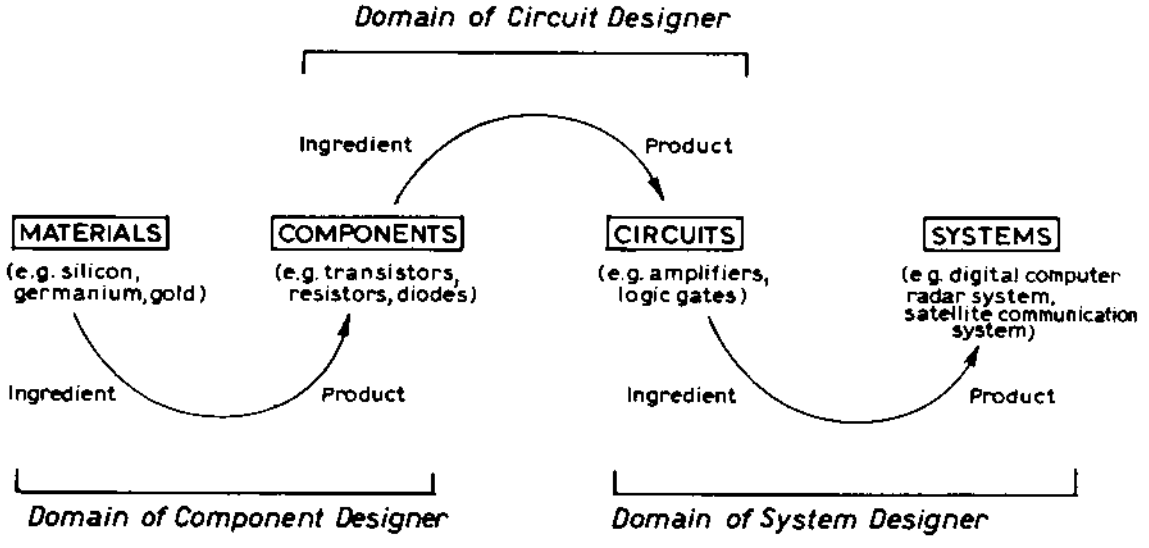


Fig. 1.3

The divisions between component, circuit and system designers are not, of course, clear-cut, and some overlap of interest is essential to communication between them. The advent of integrated circuit fabrication has, in fact, led to some merging of the functions of the component and circuit designer.

The illustration of Fig. 1.3 may also have suggested that the circuit designer is the unfortunate who is attacked from both sides, having to meet a specification laid down by the system designer with components offered by the component designer. But this would be misleading; on the one hand some knowledge of the proposed system allows him to suggest modifications to the specification and to point out inconsistencies; on the other, an acquaintance with transistor physics, for example, may enable him to provide useful feedback to the component designer concerning component fabrication. Finally, we should be aware that there is no unique starting point in the structure shown in Fig. 1.3, since invention can occur in any domain. A new component may be discovered (e.g., the transistor), a new system proposed (e.g., satellite communication) or a new circuit devised (e.g., the Eccles-Jordan bistable), and the repercussions be felt in all domains.

The transition from a collection of isolated components to a useful circuit is a creative act on the part of the circuit designer and, like all creative acts, draws heavily upon insight and previous experience. In the case of the circuit designer, such insight and experience can be enhanced considerably by a thorough study of the laws of electric circuits and of the consequences of these laws. Insight is needed, for example, in the proposal of ingenious solutions to circuit problems; it is also in evidence when, from a quick perusal of a circuit diagram, an experienced designer assesses the significant properties of the circuit. Here, then, is one important reason for a study of circuit theory.

Taken alone, of course, the designer's insight cannot be relied upon for the complete solution of any realistic design problem. At certain stages in the design process it will be necessary to confirm or explore the properties of a circuit of some complexity. The circuit, may, for example, be thought to satisfy a specification, or it may be an intermediate exploratory circuit. The circuit could, of course, be built and measured, but there are many situations in which such a course of action would be inconvenient or expensive. An integrated circuit, for example, is extremely costly if only a single copy is made. In this event the designer may choose to **simulate** circuit behaviour by means of a computer, by providing the computer with what he believes to be a satisfactory mathematical description, or model, of the circuit. By means of an algorithm the computer will then calculate the performance associated with the circuit model, the implication for the actual circuit being interpreted by the designer.

Obviously, the algorithm by means of which the computer generates the required result must be an unambiguous statement of the steps involved in the analysis of a circuit. It follows, therefore, that the associated computer program can only be written by one who understands circuit theory. Even if the circuit designer himself does not create the algorithm or program the computer, an exposure to the algorithmic approach to circuit description and analysis is invaluable in the understanding of circuit theory and hence in the development of insight.

Nevertheless, the feeling exists that the circuit designer is rarely obliged to concern himself with the steps involved in the computer analysis of circuits, and should blindly use available programs. In other words, he should supply circuit data and interpret the results he receives rather than concern himself with the precise manner in which the results were generated. But this view of the process of circuit design is unrealistic. Certainly during the interpretation of computer output the designer must confine his attention to the results of an analysis. But the competent designer must become increasingly conversant with circuit theory to understand the capabilities and limitations of existing programs, to test them effectively and use them intelligently. He may, for example, be permitted to modify or extend an existing program. What is more, with continuing improvement in man-computer communication, we are likely to see an increase in the extent of the designer's interaction with the results of a circuit analysis. He may, for example, be permitted to carry out additional and extensive circuit calculations on a conversational computer terminal. For all these reasons a good understanding of circuit theory is helpful to those involved in circuit design.

CHAPTER 2

MEASUREMENT AND MODELLING OF 2-TERMINAL COMPONENTS

The circuit designer creates a building block for the system designer by connecting together a number of components such as transistors and resistors. In order to predict the description of the resulting circuit, he must be able to describe the individual components.

By a description of a component we mean the functional relationship between the voltage(s) and current(s) associated with that component. In the present chapter we shall be concerned exclusively with components possessing only two terminals, and which are represented symbolically as in Fig. 2.1. The small circles indicate the points to which other components or measuring instruments can be connected. More fundamentally, these are the points with which an electric potential can be associated. Normally, we are concerned with, and use a voltmeter to measure, the potential difference - or voltage - between two points, so that only one voltage will be associated with a 2-terminal component.

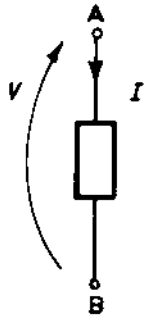


Fig. 2.1
A 2-terminal component

In order to refer to the value of a component voltage it is customary to place beside the component symbol a reference arrow, as shown in Fig. 2.1. If the potential of the terminal associated with the arrowhead is positive with respect to the other terminal then the component voltage, designated V in Fig. 2.1, is by convention said to be positive.

The presence of only two terminals also implies that only one current can be measured, since the current entering terminal A will be equal to that leaving terminal B; that this is an extremely good approximation in most circuits can be verified by measurement. Again, an arrow (Fig. 2.1) is employed to indicate the reference direction for current.

If current actually flows in the direction of the arrow its value is positive; it is negative if it flows in the opposite direction. For example, current flow in the direction B to A in Fig. 2.1 is described by a negative value of *I*.

The current and voltage reference directions can be chosen arbitrarily, since they obviously cannot influence the actual current and voltage. Nevertheless, in the interest of economy and clarity, the choice of one of the reference directions (say that for voltage) is by convention taken to imply the other reference direction (for current) in the manner shown in Fig. 2.1 (i.e., the reference arrow for current enters the terminal with which the head of the voltage reference arrow is associated). It is then only necessary to indicate either the current or the voltage reference direction.

The units of voltage and current are the volt and ampere, respectively; the latter is often abbreviated to amp. If, in the text, no unit is associated with the numerical value of a voltage or current, the units of voltage or current, respectively, are implied. Naturally, explicit mention is made of any multiples or fractions of these units (e.g., kilovolts, milliamperes).

The physical nature of a component imposes a relation between its current *I* and its voltage *V*. It is possible to associate with a component a table, such as that shown as Table 2.1, containing pairs of values of *I* and *V*. For any value of *V* there is a corresponding value of *I*, and vice versa; given one, the other is determined. More generally, we can express the current *I* as a function of the voltage *V*

Current *I* (amps)	Voltage *V* (volts)
.	.
.	.
.	.
1	0.5
2	2.7
3	5.1
.	.
.	.
.	.

Table 2.1

```
I←FV V
```

or the voltage *V* as a function of the current *I*

```
V←FI I
```

Thus, it is useful to regard the component as a function which takes either the voltage *V* or the current *I* as its argument and generates, as its result, either the current *I* or the voltage *V*, respectively. Although *FI* and *FV* are different functions, they are related; each is the inverse of the other.

The nature of the function describing a component can be found by actual measurements of its current and voltage. Such measurements are, of course, carried out in the laboratory. However, it will save a lot of time, and enable our discussions to be more effective if we study, in this book, a computer model of the component. Thus, instead of connecting a battery of known voltage to the component and observing the resulting current *I* on an ammeter (Fig. 2.2a), we employ the value of *V* as the argument of a function *FV* and display the result (Fig. 2.2b).

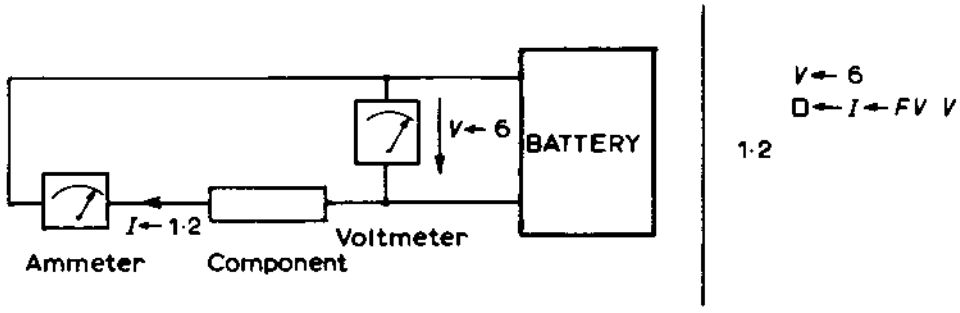


Fig. 2·2 Laboratory measurement of a component, and its computer simulation.

In this measurement the voltage is predetermined by the battery and the current is further determined by the component; we therefore refer to the voltage as the excitation of the component and the current as its response. In the computer simulation of the function we call *V* the argument and *I* its result. Although some of the measurements described in this book will undoubtedly be carried out in the laboratory, we shall henceforth be concerned solely with the computer simulation of these measurements, so that the terms excitation and argument will come to be used interchangeably, as will response and result.

Let us assume that the meters involved in the measurement of Fig. 2.2a can be read to two significant figures:

```
)DIGITS 2
```

Upon connecting a battery of 1 volt to the component, we measure a current of two amperes:

```
V←1
I←FV V
I
2
```

This single measurement tells us very little; we have identified only one point on the continuous curve which describes the relation between *V* and *I*. Far more insight into this relation is provided if the function *FV* takes a vector argument, representing a sequence of batteries of different voltages

```
      V←1  10  3  ¯4  ¯7  2
```

which are applied, one after the other, to the component. The current response associated with this vector excitation is itself a vector, and provides further examples of pairs of *I*,*V* values which satisfy the relation between *V* and *I*:

```
      I←FV V
      I
2  20  6  ¯8  ¯14  4
```

But far more insight is provided if an ordered set of equally-spaced values is chosen for the voltage excitation vector *V* for then the pattern discernible in the current response vector *I* shows that the relation between *I* and *V* is linear; that it is also homogeneous can easily be tested:

```
      V←⍳10
      V
1 2 3 4 5 6 7 8 9 10
      I←FV V
      I
2 4 6 8 10 12 14 16 18 20
      FV 0
0
```

The linearity of this relation (we speak also of the linearity of the function and of the component) can further be demonstrated by the element-by-element division of *I* and *V*

```
      I÷V
2 2 2 2 2 2 2 2 2 2
```

and also by calculating the difference between successive values of current (i.e., successive elements of *I*) generated by equally-spaced values of voltage (i.e., elements of *V*). For the latter purpose we define a function *DIFF*

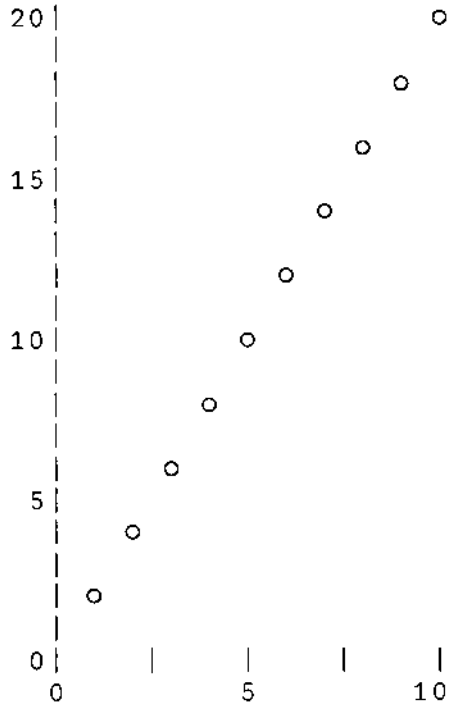
```
    ∇ R←DIFF X
[1]   R←(1↓X)-¯1↓X
    ∇
```

which generates the difference between successive elements of its vector argument. Applied to the vector I it shows that a 1 volt increase in V, for any value of V, causes a 2 ampere increase in I:

```
      DIFF I
2 2 2 2 2 2 2 2 2
```

As expected with a linear homogeneous function, the ratio of voltage and current is equal to the ratio of changes in these quantities. This is particularly evident from a plot of the I,V pairs we have generated:

```
      30 PLOT I VS V
```



For the ranges of V and I that we have investigated it follows that the relation between these quantities can be expressed as

$$I \leftarrow G \times V \qquad (2.1)$$

where G is called the conductance of the component, and here has the value 2; that is $G \leftarrow 2$. If I and V in equation (2.1) are expressed in amps and volts respectively, the conductance is measured in mhos. If the unit of a conductance is left unspecified, it should be assumed to be the mho.

Expression (2.1) is referred to as Ohm's Law. Thus, the physical processes which determine the current due to a voltage *V* are simulated, in the computer, by the multiplication of *V* by the constant *G*.

The conductance can be regarded either as the ratio of the current and the corresponding voltage, or as the ratio of corresponding changes in these quantities, since these ratios are identical. Indeed, to emphasize the fact that conductance is the ratio of current and voltage, and at the same time permit the choice of voltage excitations separated by a value other than one volt, the differencing function can be applied to both *I* and *V* in computing the ratio of changes in these quantities:

```
      V←3×⍳10
      I←FV V
      (DIFF I)÷DIFF V
2 2 2 2 2 2 2 2 2
```

From actual or simulated measurements we have deduced the functional relation

```
      I←2×V
```

between *I* and *V*, and can incorporate this relation in our own derived model of the component by defining the function *FVMODEL*:

```
    ∇ R←FVMODEL V
[1]   R←2×V
    ∇
```

The equivalence of functions *FV* and *FVMODEL* can be explored as thoroughly as desired. For example:

```
      V←1 2 3 4 5 6 7 8 9 10

      FV V
2 4 6 8 10 12 14 16 18 20
      FVMODEL V
2 4 6 8 10 12 14 16 18 20

      ∧/(FV V)=FVMODEL V
1
```

To make the idea of a computer model of a component embodied in an APL function more familiar, we set out below the corresponding activities in the laboratory and on the computer:

	Computer	Laboratory
1.	Systematically evaluate a function simulating an electrical component.	Make a systematic series of measurements of a component.
2.	From the pattern associated with the results of the previous step, deduce a linear and homogeneous relation.	
3.	Propose a mathematical model of the relation between V and I.	
4.	Test the mathematical model against function evaluations.	Test the mathematical model against laboratory measurements.

Of particular importance in measurement is systematic exploration leading to a pattern of response from which a relation, and hence a model, can be deduced.

2.2 THE INVERSE FUNCTION

The measurements described above employed, as excitation, a device (battery) whose voltage appeared to be the same irrespective of what it was connected to. We measured the current flowing in response to this excitation. In contrast, we could propose - at least on paper - a situation in which the roles played by voltage and current are reversed. In Fig. 2.3 we employ as excitation some hypothetical device which we again call a battery, and which delivers the same current irrespective of what it is connected to. This excitation is applied to the same component as before with the result that a voltage appears between its terminals; this response voltage is measured on a voltmeter. The computer simulation of this measurement is also shown in Fig. 2.3.

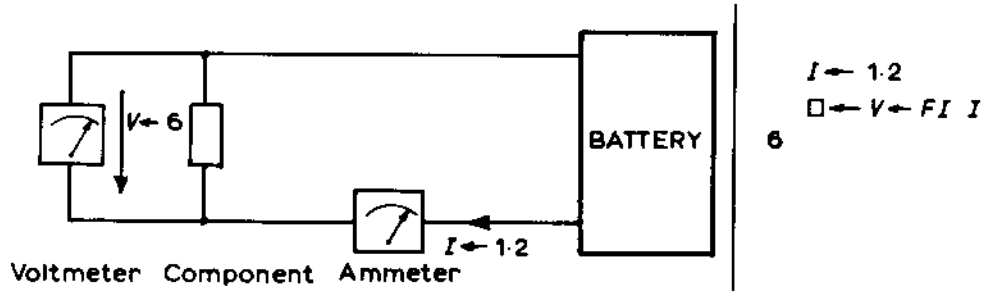


Fig. 2·3 Laboratory measurement of a component, and its computer simulation.

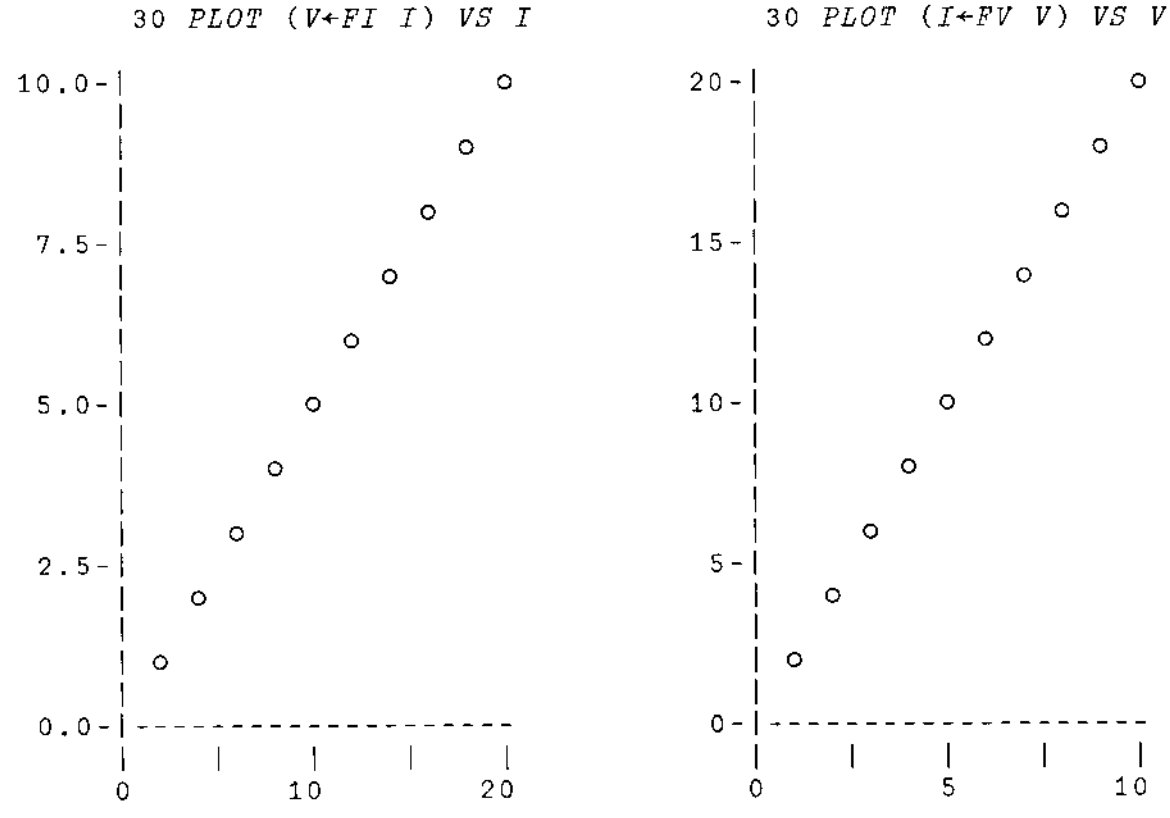
If we employ as excitation the current vector *I* which was previously the response, the response is now a voltage vector *V*

```
      I
2 4 6 8 10 12 14 16 18 20
      V←FI I
      V
1 2 3 4 5 6 7 8 9 10
```

identical with the vector of voltages previously employed as excitation. In this way we have verified that, for the values of *I* and *V* considered, *FV* and *FI* are inverse functions; i.e., the result of one is negated by the other:

```
      ∧/V=FI FV V
1
      ∧/I=FV FI I
1
```

The inverse relation is particularly evident if plots of *FI* and *FV* are examined and it is noted that one can be obtained from the other by the interchange of ordinate and abcissa:



The inverse relation between *FV* and *FI* leads directly to a model of the function *FI*. This function is linear and homogeneous,

$$V \leftarrow R \times I \qquad (2.2)$$

and described by a single parameter called the resistance of the component. Like (2.1), equation (2.2) is also referred to as Ohm's Law. If *V* and *I* in equation (2.2) are expressed in volts and amperes respectively, the unit of resistance is the ohm. That the value of *R* is 0.5 ohms is evident from application of the differencing function to successive values of *V* generated from equally-spaced values of the argument *I*:

```
      (DIFF V)÷DIFF I
0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5
```

a calculation which accomodates any spacing of the elements of *I*, and which emphasizes the fact that resistance is the ratio of voltage and current. From this result it follows that *R*←0.5. It is a simple matter to define a function *FIMODEL* and to test it. However, since *FV* and *FI* are mutually inverse, there must be some relation between the two parameters involved, the conductance *G* and the resistance *R*. Clearly, the relation is

```
      R←÷G
```

or

```
      1=R×G
```

2.3 NONLINEAR COMPONENTS

Let us explore the characteristic of a new component described by the function *ONEV*1*. The voltage excitation

```
      V←¯6+⍳11
```

gives rise to a current response vector

```
      I←ONEV1 V
      I
¯10 ¯8 ¯6 ¯4 ¯2 0 2 4 6 8 10
```

of such a form that the function *ONEV*1 is apparently linear and homogeneous, and characterized by a conductance whose value is derived from use of the differencing function:

```
      (DIFF I)÷DIFF V
2 2 2 2 2 2 2 2 2 2
```

---

* The structure of this function name is partly in keeping with the convention adopted by semiconductor component manufacturers, where the first number of a component name (e.g., 2N1304) is one less than the number of terminals, and the last number serves to identify the component.

Thus, the function *ONEV*1 is apparently no different from the function *FV* explored earlier.

But let us now increase the range of voltage excitation over which we explore the function *ONEV*1. A series of voltage excitations ranging from ¯45 to 45 volts

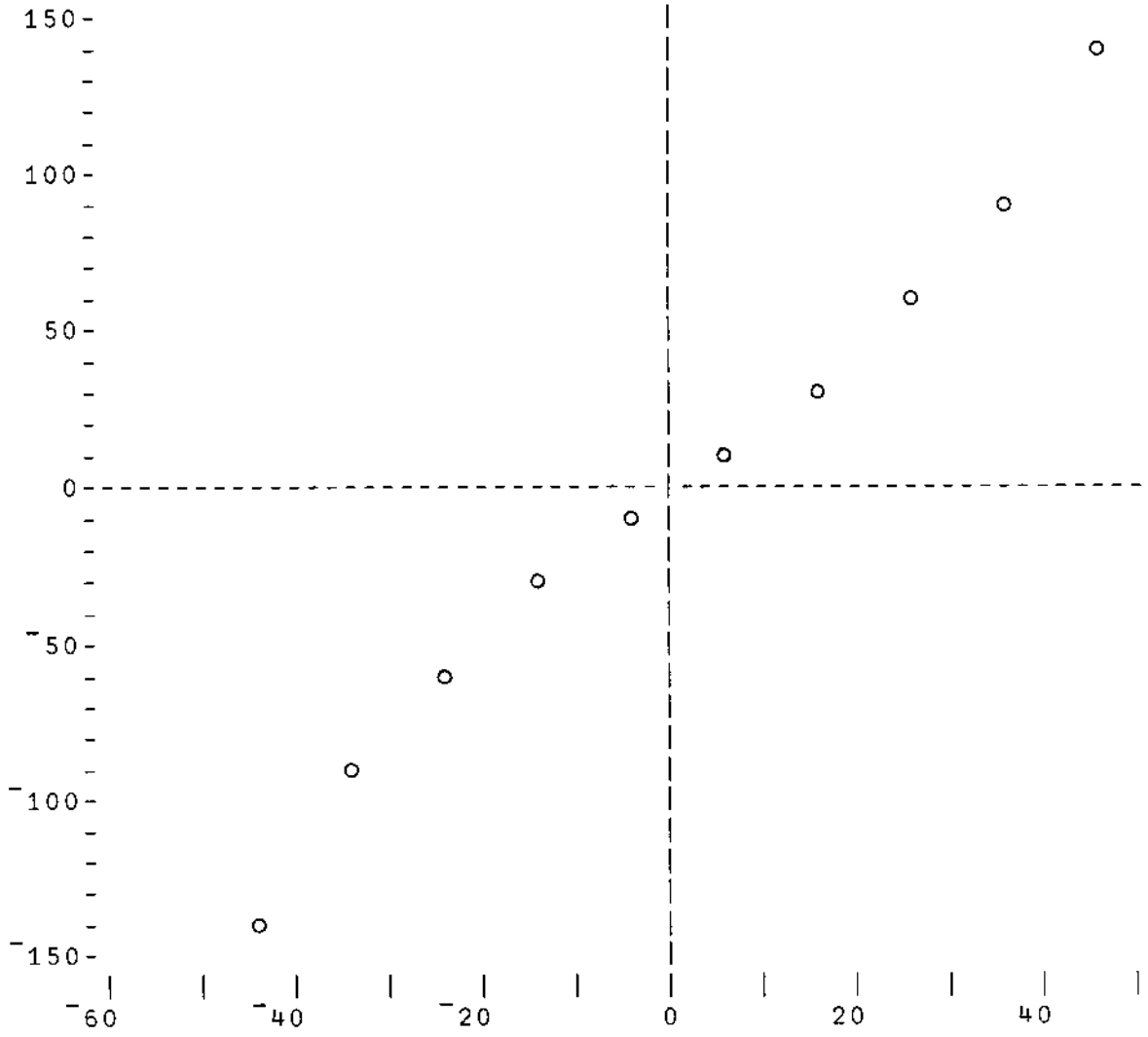
```
      V←¯55+10×⍳10
```

leads to a vector of current response

```
      I←ONEV1 V
      I
¯1.4E2 ¯91 ¯58 ¯32 ¯10 10 32 58 91 1.4E2
```

such that a plot of current versus voltage reveals a property of the component not previously suspected:

```
      30 60 PLOT I VS V
```



The apparently linear relation between *I* and *V* is only an approximation valid over a restricted range, and *I* is a nonlinear function of *V*. We say that *ONEV*1 is a nonlinear function, and we refer to a nonlinear component. A further increase in the range of excitation voltage would reveal more detail of the function *ONEV*1.

What comments can usefully be made about the component simulated by the function *ONEV*1? The important point, for the engineer, is the use to which the component will be put. If the approximately linear relation between *I* and *V* over a limited range of excitation is the aspect which is exploited, either of the relations

```
I←2×V
V←.5×I
```

is appropriate. On the other hand a particular application may exploit the curvature of the current-voltage characteristic. In this case it may be desirable to obtain some mathematical model of the function *ONEV*1 which describes the departure from linearity.

2.4 CURVE FITTING

Consider a two-terminal component simulated by a new function *F*1 which, like *FV* and *ONEV*1, takes the value of the component voltage as its single argument. An excitation voltage vector whose elements are spaced by one volt, and which includes the value zero,

```
      □←V←0,⍳7
0 1 2 3 4 5 6 7
```

leads to a vector of current response

```
      □←I←F1 V
0 2.2 4.8 7.8 11 15 19 24
```

whose elements are not equally spaced:

```
      DIFF I
2.2 2.6 3 3.4 3.8 4.2 4.6
```

However, a further application of the differencing function shows that the second differences

```
      DIFF DIFF I
0.4 0.4 0.4 0.4 0.4 0.4
```

are equal. The vectors *I*, *DIFF I*, and *DIFF DIFF I* are sufficient to deduce the nature of the function *F*1 and to

define a model in which *I* is a polynomial in *V*. The procedure is simple: element-by-element division of the first elements of the above three vectors

```
      A←0 2.2 0.4
```

by the vector

```
      B←!0 1 2
```

yields

```
      A÷B
0 2.2 0.2
```

a vector whose elements are the coefficients involved in the following expression for *I*:

```
      I=0+(2.2×V)+0.2×V×V-1
```

That this expresses the relation between *I* and *V* can be investigated by defining the corresponding model of the component

```
    ∇ I←F1MODEL V
[1]   I←(2.2×V)+(0.2×V×V-1)
    ∇
```

and testing it against the original function:

```
      ∧/(F1 V)=F1MODEL V
1
```

This method of fitting a polynomial expression to a set of function evaluations may well have been encountered previously by the reader as a topic in elementary algebra*, and will not be proved here.

By interchanging the ordinate and abcissa of the plot of *I* versus *V* it is clear that, given a value of *I*, we can work backwards to the corresponding value of *V*; in other words, an inverse function exists. However, there is no simple general expression for the inverse of a polynomial and therefore it is not a straightforward matter to define a function which is the inverse of *F1*. Nevertheless, as we shall see in chapter 10, it is possible, quite simply, to generate specific numerical inverses of a function such as *F1*.

---

* Iverson, K. E., Algebra: an algorithmic treatment, Addison-Wesley, Reading, Mass., 1972.

2.5 LOCAL CONDUCTANCE

The concepts of conductance and resistance may still be applied, though with care, to a nonlinear component. For the component described by the function *F1*, the element-by-element division of a current response vector by the vector of voltage excitation

```
      V←⍳10
      I←F1 V
      I÷V
2.2 2.4 2.6 2.8 3 3.2 3.4 3.6 3.8 4
```

yields, as expected with a nonlinear function, a vector whose elements are not all equal. However, for a nonlinear component, the concept of conductance as the ratio of absolute values of current and voltage does not find extensive application.

A different and far more useful measure of conductance is afforded by considering changes in the component voltage and current. Application of the differencing function to a voltage excitation vector

```
      V←0.5×⍳10
      DIFF V
0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5
```

yields a vector whose elements are the changes in voltage between successive excitations. A similar application of the differencing function

```
      I←F1 V
      DIFF I
1.15 1.25 1.35 1.45 1.55 1.65 1.75 1.85 1.95
```

yields the changes in current corresponding to successive changes in voltage excitation. Element-by-element division

```
      (DIFF I)÷DIFF V
2.3 2.5 2.7 2.9 3.1 3.3 3.5 3.7 3.9
```

now generates a vector whose elements are the ratio of corresponding changes in current and voltage. For this reason the elements are termed the differential conductance or incremental conductance of the component. For example, the change in current response due to a change in voltage excitation from 2 to 2.5 volts is equal to 2.9×0.5 amps. In other words, a statement that the differential conductance of a component is 2.9 mho at 2.25 volts could be interpreted as

```
      ΔI←2.9×ΔV
```

where ΔI and ΔV are increments in current and voltage about the values associated with *V*←2.25. The above equation is

associated with a specific value of V because the differential conductance so defined is a function of the values of I and V about which the changes ΔI and ΔV occur. Thus, the same increment in voltage excitation, but from 4.5 to 5.0 volts, now leads to a different change (3.9×0.5 amps) in current response.

The limitation of the above equation is that prior knowledge is required of the increment in V (0.5 volts) for which it is valid. To avoid this drawback we note that in the limit as the changes in current and voltage approach zero, the ratio of these changes approaches the tangent of the function describing the component. It is to the tangent of the function that we conventionally assign the name differential or incremental conductance. Thus, a more accurate calculation of the differential conductance associated with a value of V of 2.5 volts is

```
      V←2.5 2.501
      I←F1 V
      (DIFF I)÷DIFF V
3
```

and describes the relation

$$\Delta I \leftarrow 3 \times \Delta V \qquad (2.3)$$

The fact that this <u>linear, homogeneous</u> relation is associated with a nonlinear function $F1$ is, of course, a consequence of the fact that, over a small enough range, any continuous function can be approximated sufficiently accurately by its tangent. Because equation (2.3) is applicable over only a small range, the conductance so defined is often referred to as the <u>local conductance</u>.

2.6 THE SEMICONDUCTOR DIODE

Many of the nonlinear components encountered in electronic circuits are manufactured from material (germanium or silicon) known as a <u>semiconductor</u>. Those having two terminals are known as <u>semiconductor diodes</u>. One such diode* is simulated by the function *DIODEV*. Let us examine its characteristic by choosing values of excitation voltage within the relatively small range from ¯1 to 1 volt

```
      V←.2×¯6+⍳11
```

---

* The name <u>diode</u> strictly applies to a component possessing <u>two</u> terminals. Nevertheless, the name has come to be used almost exclusively to describe components having a markedly nonlinear voltage-current characteristic similar in form to that of the semiconductor diode. Since such components find extensive use in circuits known as rectifiers, they are often referred to as rectifier diodes.

and assuming that the response current can be observed to an accuracy of three significant figures:

```
      )DIGITS 3
```

The vector of response currents

```
      □←I←DIODEV V
¯1E¯11 ¯1E¯11 ¯1E¯11 ¯1E¯11 ¯1E¯11 0 2.98E¯8 8.89E¯5 0.265
      790 2.35E6
```

extends over a very wide range of values. Three aspects of this current response vector should be noted:

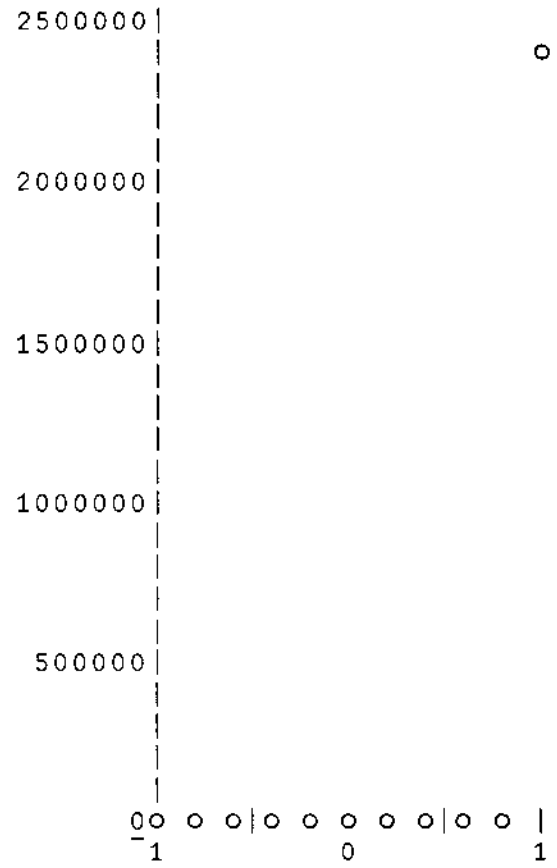
(1) For $V=0$, $I=0$,

(2) When V is negative a very small but constant current flows,

(3) As V increases from zero, I increases very rapidly; roughly, its value is multiplied by the same amount for equal increments in V.

A plot of I versus V emphasises the fact that the relation between I and V is highly nonlinear:

```
      30 30 PLOT I VS V
```



This plot shows that the component appears to strongly resist the flow of current in one direction while permitting high currents to flow at low voltage in the opposite direction. This property is extensively exploited in electronic circuit design, and is implied by the symbol (Fig. 2.4) employed for a semiconductor diode; the direction of the arrow incorporated in the symbol indicates the 'easy' direction of current flow.

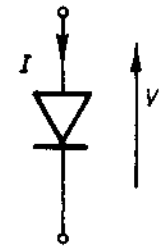
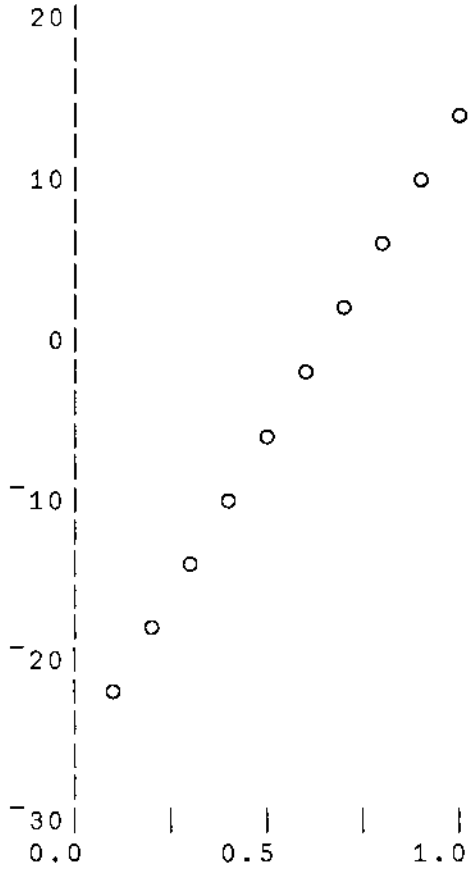


Fig. 2.4

An attempt to discover a concise polynomial relation between *I* and *V* will be defeated by the fact that even the tenth application of the differencing function fails to establish a vector of equal elements. A more successful approach is to recall property (3) above, which suggests that the function may be strongly exponential. A plot of the natural logarithm of *I* versus *V* for positive values of *I* tends to confirm our expectation:

```
V←.1×⍳10
I←DIODEV V
30 PLOT (⍟I) VS V
```



However, property (1) prevents us from concluding that the relation is of the form

```
I←W×*K×V                                                    (2.4)
```

over the entire range of voltage excitation explored. Nevertheless, before proceeding further, we can tentatively establish the value of *K*

```
      (DIFF ⍟I)÷DIFF V
40.18 40 40 40 40 40 40 40 40
```

From this result we conclude that relation (2.4) may be an accurate model of the function, for $V>0.1$, if *K* is equal to 40 reciprocal volts. For this range of *V*, *I* is much greater than unity, so that a possible modification of (2.1) to accomodate property (1) is

```
I←W×¯1+*40×V                                                (2.5)
```

The value of *W* may tentatively be found by observing that it is, to a very good approximation, the magnitude of *I* for any value of *V* more negative than about ¯0.1 volt. Examination of the current response vector *I* shows that this value is 1*E*¯11. We therefore define a function to model *DIODEV*

```
    ∇ I←DIODEVMODEL V
[1]   I←1E¯11×¯1+*40×V                                      (2.6)
    ∇
```

and test it with the original voltage excitation vector

```
      V←0.2×¯6×⍳11
      ∧/(DIODEV V)=DIODEVMODEL V
1
```

Clearly we have devised a model which is acceptable over the range of *V* explored. The model now helps us to appreciate the electrical properties of the semiconductor diode. A diode described by a relation of the form (2.6) will be called an **exponential diode**.

A number of comments are relevant at this point. First, the constant *W* is called the **saturation current** of the diode and is normally denoted by *IS*. Second, practical experience with a number of different semiconductor diodes will gradually reveal the fact that, whereas *IS* exhibits large variations from one diode to another, the constant (here 40) by which the diode voltage is first multiplied tends to be invariant. Both these properties can be explained, quite straightforwardly, by a consideration of the physical mechanisms underlying the behaviour of a semiconductor diode. Although such considerations lie outside the scope of this book, it is very relevant to the task of measurement and modelling to remark that the circuit

designer's collaboration with the device designer will result in the former being armed with knowledge of the theoretical relation between *I* and *V*, so that his chances of obtaining a good, concise model are enhanced. The knowledge will also assist him practically; in the present example he will be forewarned against applying an excitation voltage which is so high as to cause damage to the component and which, with the function *DIODEV*, generates a result lying outside the capacity of the computer

```
      DIODEV 6
DOMAIN ERROR
DIODEV[1] R←1E¯11×¯1+⋆40×V
                       ∧
```

Even with prior knowledge concerning the physical mechanism of the component being modelled, it is unlikely in practice that a concise polynomial or exponential relation can result in an **exact** model of the component, even over a limited range of excitation. In this event a test of the form

```
      ∧/(DEVICE V)=DEVICEMODEL V
```

will rarely be successful; normally, some acceptable response error (or 'measure of goodness') will have to be defined (say, *E*←0.1) before any decision is made as to whether the model is acceptable:

```
      ∧/E>|(DEVICE V)-DEVICEMODEL V
1
```

In case our study of the polynomial approximation to a function suggested that nonlinearity is inevitably associated with the absence of a general inverse relation, it is valuable to see if an inverse relation can be found for the exponential diode. Examination of (2.6) shows that this is possible, and the appropriate function

```
    ∇ V←DIODEVINV I
[1]   V←(⍟1+I÷E¯11)÷40
    ∇
```

can easily be tested

```
      ⎕←V←.1×¯1+⍳11
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
      ∧/V=DIODEVINV DIODEV V
1
```

Thus, a nonlinear function may well possess an inverse for which a concise and general expression exists.

Finally, we return to the remark that the semiconductor diodes are described, theoretically, by a relation of the form

```
I←IS×¯1+*40×V
```

In this situation we may observe, and exploit, the fact that current is a **linear** function of a quantity - let us call it *X* - which is a **nonlinear** function of *V* according to the relation

```
X←¯1+*40×V
```

The definition of a function *TRP* which transforms *V* according to this relation

```
    ∇ R←TRP V
[1]   R←¯1+*40×V
    ∇
```

allows the equivalent values of *X* to be generated first, so that the constant (*IS*) associated with the **linear** relation between *I* and *X* can be explored and modelled by the methods appropriate to linear functions. For the component *DIODEV* and the excitation previously employed,

```
      V
¯1 ¯0.8 ¯0.6 ¯0.4 ¯0.2 0 0.2 0.4 0.6 0.8 1

      ⎕←X←TRP V
¯1 ¯1 ¯1 ¯1 ¯1 0 2.98E3 8.89E6 2.65E10 7.9E13 2.35E17

      I←DIODEV V
      I÷X
1E¯11 1E¯11 1E¯11 1E¯11 1E¯11 1 1E¯11 1E¯11 1E¯11 1E¯11
      1E¯11
```

This result confirms the form of the function *DIODEV*, and establishes that the value of *IS* is 1E¯11. We shall adopt this approach again in Chapter 6 when attempting to model a three-terminal nonlinear device.

In what follows we shall encounter both nonlinear and linear homogeneous components. However, for the sake of brevity we shall refer to the latter merely as linear components, or, simply, resistances or conductances, and employ the symbol shown in Fig. 2.5. The number associated with the symbol denotes the value of the conductance (in mho, unless otherwise stated) associated with the component.

Fig. 2.5

CHAPTER 3

COMPONENT INTERCONNECTION

The circuit designer is ultimately interested in the interconnection of quite a large number of components. In order to predict the properties of such an interconnection he must know, in addition to the voltage-current relation for each component, the nature of any constraints that are imposed upon the currents and voltages by the very act of interconnection.

3.1 CURRENT CONSTRAINTS

Fig. 3.1 shows an interconnection of some 2-terminal components, each of which is assigned a reference number as well as a current reference direction. Some may be batteries, some linear resistors and others nonlinear diodes; since the nature of each is immaterial they are all represented by the same symbol. Ammeters record the currents $I1, I2, \ldots, I11$ flowing in the components; the numerical value of each current is shown in the figure.

Examination of the current values yields some interesting facts. For example, the current leaving component 4 is equal to the current entering component 6. Since no other component is connected to the junction of components 4 and 6, this is not surprising in view of what we know about current and charge. Current is the rate of flow of charge past a point in an electrical circuit. Because charge cannot accumulate at a point, the rate of supply of charge to the junction of components 4 and 6 must equal the rate at which charge is removed from this junction. In other words, $I4=I6$

The same principle - that the net rate of supply of charge to (or the current into) a junction point is zero - is seen to apply to junctions of more than two components. For example, examination shows that $0=I11+I7+I5-I2$ and the same situation will be found to apply to all junction points. This experimentally observed fact is embodied in Kirchhoff's Current Law:

> The algebraic sum of the currents flowing into a junction is zero.

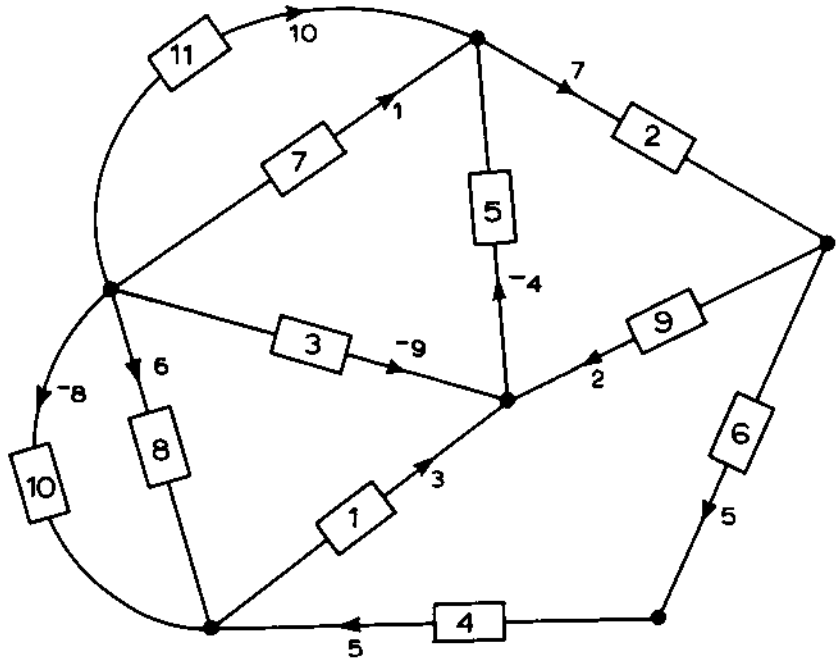


Fig. 3.1
An interconnection of numbered components, with the measured currents, in amperes, shown.

Kirchhoff's current law, in fact, constitutes a constraint upon the currents flowing in the components. Thus, as soon as one terminal of each of components 2, 5, 7 and 11 are connected together, the constraint $0=I11+I7+I5-I2$ is imposed; it is imposed solely by the interconnection and is independent of the nature of the components connected together: component 2 may be a battery, 5 and 11 linear resistors and 7 a nonlinear diode, or all four may be nonlinear diodes.

It is possible to effect an economy of notation which will ease the task of discussing circuits containing a large number of components. Rather than refer to the currents $I1$, $I2$, $I3,\ldots,I11$ whose numerical values are shown in Fig. 3.1 we may refer instead to a single current <u>vector</u> I whose elements are $I[1]$, $I[2]$, $I[3],\ldots,I[11]$. Thus, the currents flowing in the circuit of Fig. 3.1 are described by a single vector

```
      I←3 7 ¯9 5 ¯4 5 1 6 2 ¯8 10
```

Indexing is then all that is required to discover the current in any one branch

```
      I[3]
¯9
```

or any set of branches

```
      I[1 3 5 9]
3 ¯9 ¯4 2
```

Note that, in contrast to the previous chapter, the vector *I* does not describe a sequence of values of the same current, but rather a number of coexisting currents.

It is important to note, for components sharing a common junction (e.g., components 1, 3, 5 and 9), that a simple sum reduction over the corresponding elements of *I*

```
      +/I[1 3 5 9]
¯8
```

does not necessarily yield a zero result. Kirchhoff's current law requires that the direction of each current be taken into account. We must therefore first multiply *I* by a vector whose elements are 1 or ¯1 according to whether the reference direction for each component is into or out of the common junction respectively; a zero multiplier is used for those components which are not connected to the junction. For the previous example (components 1, 3, 5 and 9) the appropriate vector is

```
      A←1 0 1 0 ¯1 0 0 0 1 0 0
```

and the result of element-by-element multiplication by *I* is

```
      A×I
3 0 ¯9 0 4 0 0 0 2 0 0
```

The sum reduction of the result is zero as required by Kirchhoff's current law:

```
      +/A×I
0
```

A useful generalization of Kirchhoff's current law is revealed if we consider the total current flowing out of (or into) a closed region, such as the one indicated by a (dashed) curve in Fig. 3.2 which is a repeat of the circuit of Fig. 3.1. The total current flowing into this surface is zero:

```
      +/I×1 1 1 0 ¯1 ¯1 0 0 0 0 0
0
```

This is commensurate with the requirement that charge shall not accumulate at any point in a circuit, and is an example of the property of Kirchhoff's current law that it applies not only to simple junctions but also closed curves (a sort of generalized junction).

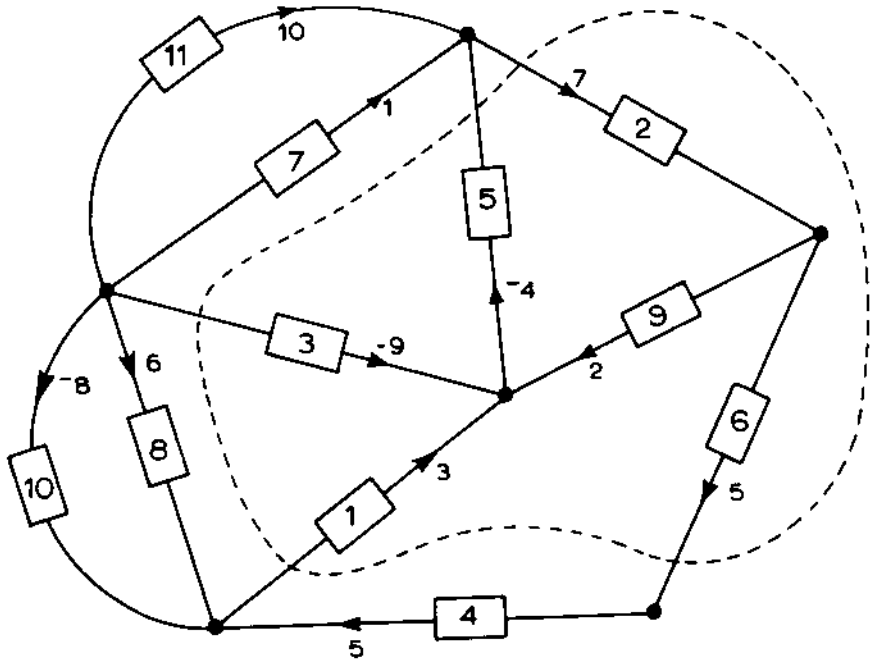


Fig. 3.2
The circuit of Fig. 3.1, and an arbitrarily drawn closed curve (dashed).

3.2 VOLTAGE CONSTRAINTS

The interconnection of components also places a constraint upon their voltages. In Fig. 3.3a the circuit of Fig. 3.1 is again repeated, but now the voltages measured by a voltmeter across each component are indicated. We straightway adopt the notion of a voltage vector V whose elements are the voltages across the appropriately numbered components.

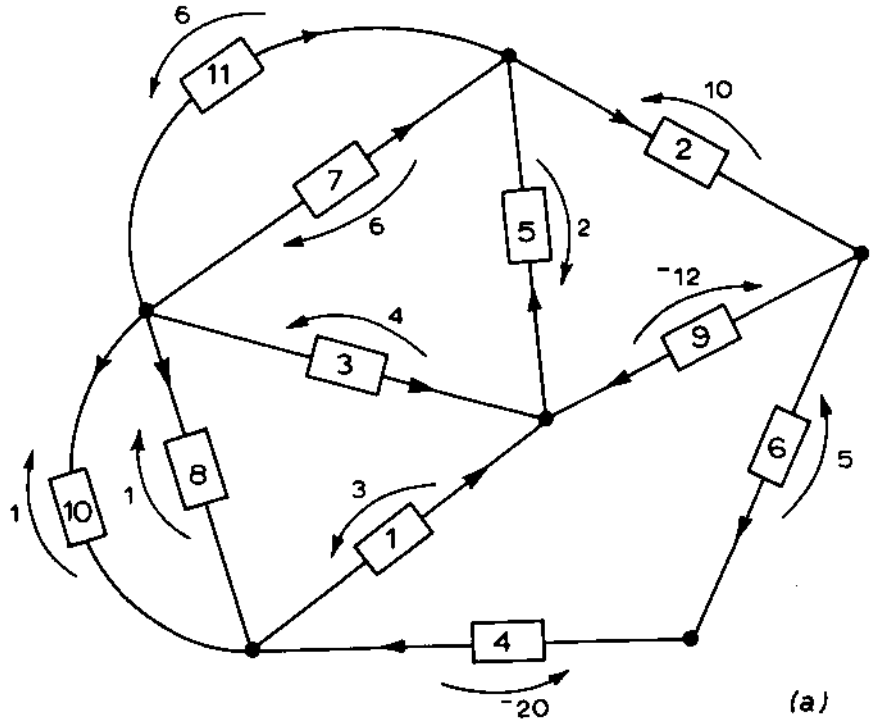
```
V←3 10 4 ¯20 2 5 6 1 ¯12 1 6
```

The potential difference or voltage across a component is the difference between the potentials associated with its terminals, due account being taken of the reference direction assigned to the component voltage. Thus, if the junctions between components are numbered as shown in Fig. 3.3b, the voltage of component 5 is the potential of junction 2 minus the potential of junction 3.

It is only possible to associate a potential with any junction with respect to a reference potential (a good analogy is height above sea-level). The reference potential is arbitrary. If junction 1 is selected as this reference point, it will be found that the potential vector

```
P←0 12 10 15 16 ¯5
```

describes the potentials of the six junctions; the first element of p is zero, consistent with the choice of junction 1 as the potential reference. The voltage of component 5,



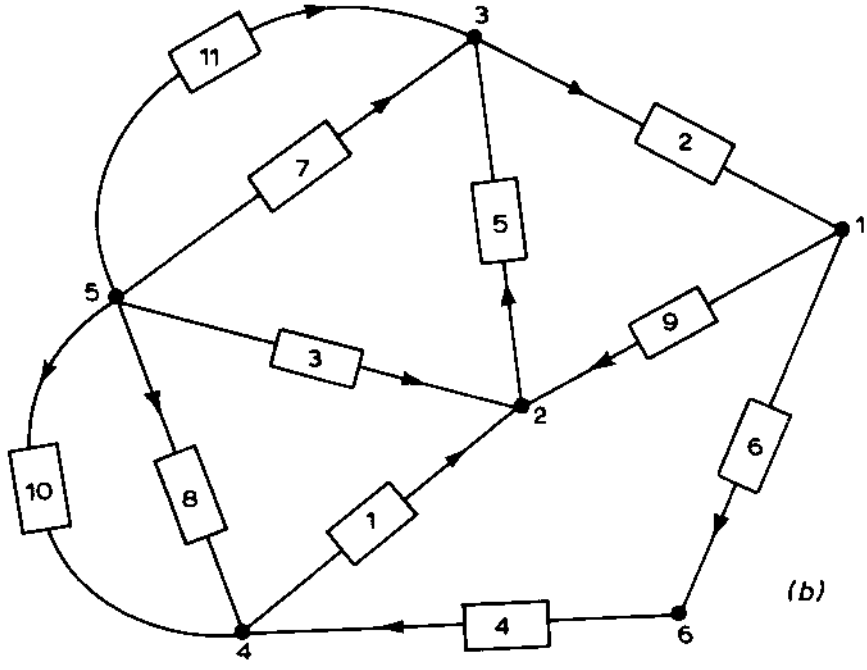


Fig. 3·3 The circuit of Fig. 3·1, with (a) the values of measured component voltages shown, and (b) the node numbers shown.

connected to junctions 2 and 3, is a simple function of the potential vector *P*:

```
      -/P[2 3]
2
```

The same result can be obtained by first forming a vector

```
      B5←0 1 ¯1 0 0 0
```

whose elements are 1, ¯1 or 0 respectively, according to whether the corresponding junction is at the positive or negative reference point of the component, or not associated with the component at all. Element-by-element multiplication by the potential vector *P*, followed by sum reduction, then yields the voltage of component 5

```
      V[5]=+/P×B5
1
```

To emphasize the fact that the voltage of a component is a function only of its terminal potentials, we define a function which takes the potential vector as its single argument and returns the voltage of component 5:

```
    ∇ V←VOLT5 P
[1]   V←+/P×0 1 ¯1 0 0 0
    ∇
      VOLT5 P
2
```

It is now easy to demonstrate the arbitrary nature of the choice of reference point for potential as far as component voltages are concerned. The choice of junction 5, whose potential is 16 volts with respect to junction 1 (the previous reference point) leads to a new potential vector

```
      □←P←P+¯16
¯16 ¯4 ¯6 ¯1 0 ¯21
```

but one which yields the same values for the voltage of component 5:

```
      VOLT5 P
2
```

It is immediately obvious that two components connected in parallel (i.e., which are connected across the same two junctions) have the same voltage. Thus, with reference to components 8 and 10

```
      V[8]=V[10]
1
```

or

```
      V[8]-V[10]
0
```

But the equality of voltages associated with components connected in parallel is only a special case of a more general property. Examination of the voltages associated with components 1, 3 and 8 which form a closed loop within the circuit shows that, if we assign a positive reference to traversal of the loop in one direction (e.g., clockwise), then the sum of the component voltages - taking the reference direction into account (Fig. 3.4) - is zero:

```
      +/V[1 3 8]×1 ¯1 1
0
```

The voltage V[3] of component 3 was multiplied by ¯1 because its reference direction opposes the chosen reference direction for the loop. Choice of the opposite (anti-clockwise) loop reference direction

```
      +/V[1 3 8]×¯1 1 ¯1
0
```

is equally valid. Total disregard for reference directions, however,

```
      +/V[1 3 8]
8
```

will in general yield a non-zero result which is not meaningful. The equality of V[8] and V[10] is a special case of the voltages around a loop summing to zero.

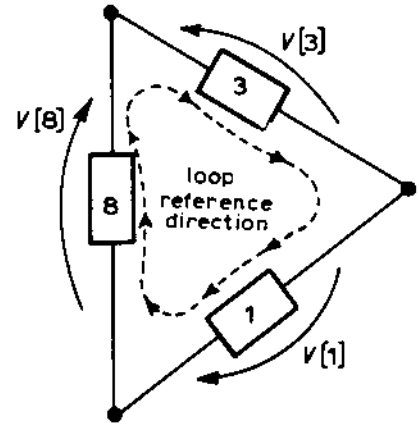


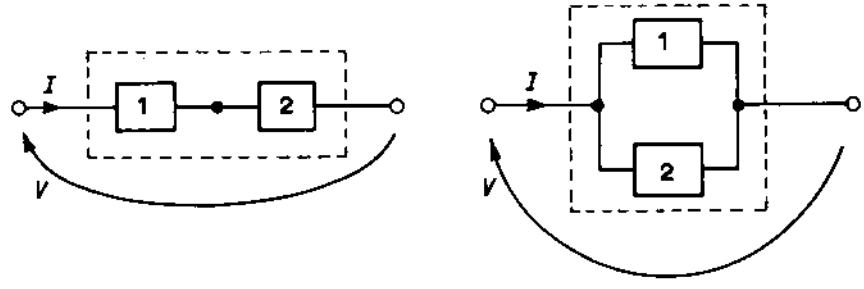
Fig. 3·4

In the circuit of Fig. 3.3a a large number of loops can be identified, and in each case the sum of the voltages around the loop, taking proper account of reference directions, is zero. This experimentally observed result is summarized by Kirchhoff's Voltage Law:

> The algebraic sum of the voltages around any loop in a circuit is equal to zero.

from which it is clear that the interconnection of components places constraints upon their voltages as well as their currents. The constraint embodied in Kirchhoff's voltage law is in accordance with our understanding of electric potential. The work that must be done on a unit charge to move it from one point to another is equal to the difference in potential between (i.e., the voltage between) these two points. The fact that movement of a unit charge from one point back to the same point involves no net work, irrespective of the path taken, is commensurate with a total voltage of zero around the loop so defined. This is analogous to saying that whatever rises and falls we have in a journey which ends at its starting point, they must sum algebraically to zero.

There are two useful ways of interconnecting two 2-terminal components to form a new component, and they are shown in Fig. 3.5. Part (a) is called the series connection and part (b) the parallel connection of two components. The use of dashed lines indicates, in each case, that we wish to determine the relation between the current I and voltage V of the new 2-terminal component that has been formed. In Fig. 3.5a the internal connection between components 1 and 2 is indicated by a solid dot to signify that this point is inaccessible. If it were accessible, Fig. 3.5a would represent a three-terminal component.



(a) series connection of two components

(b) parallel connection of two components

Fig. 3.5

3.3 SERIES CONNECTION

Consider the series connection first (Fig. 3.6). Without knowing the nature of the two components 1 and 2 we can draw two conclusions by invoking Kirchhoff's Laws. The current law requires that

$$IS \leftarrow I[1] \leftarrow I[2]$$

since no other component is connected to the junction of components 1 and 2. The voltage law requires that

$$VS \leftarrow V[1] + V[2]$$

To define the voltage-current relation of the new component of Fig. 3.6, it is necessary to eliminate $I[1]$, $I[2]$, $V[1]$ and $V[2]$, variables that are associated with the components internal to the new component, from the above two expressions. The way in which this is achieved is best shown by example.

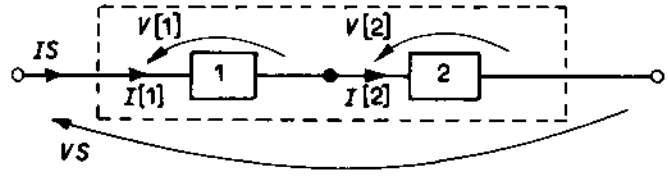


Fig. 3·6

Suppose that component 1 is a linear resistor of 5 ohms, and that component 2 is an exponential diode having a saturation current of $1E^{-}11$ amperes, so that their separate voltage-current characteristics are as shown in Fig. 3.7a-b. The constraint $IS \leftarrow I[1] \leftarrow I[2]$ allows the characteristics to be plotted to a common base (Fig. 3.7c) which can now be labelled IS (the current of the new component). The constraint $VS \leftarrow V[1]+V[2]$ allows V, (the voltage of the new component) to be obtained simply by adding the ordinates $V[1]$ and $V[2]$ associated with specific values of the abscissa $IS(\leftarrow I[1] \leftarrow I[2])$. In this way the relation between VS and IS can be obtained by graphical means, irrespective of whether the components 1 and 2 are linear or nonlinear.

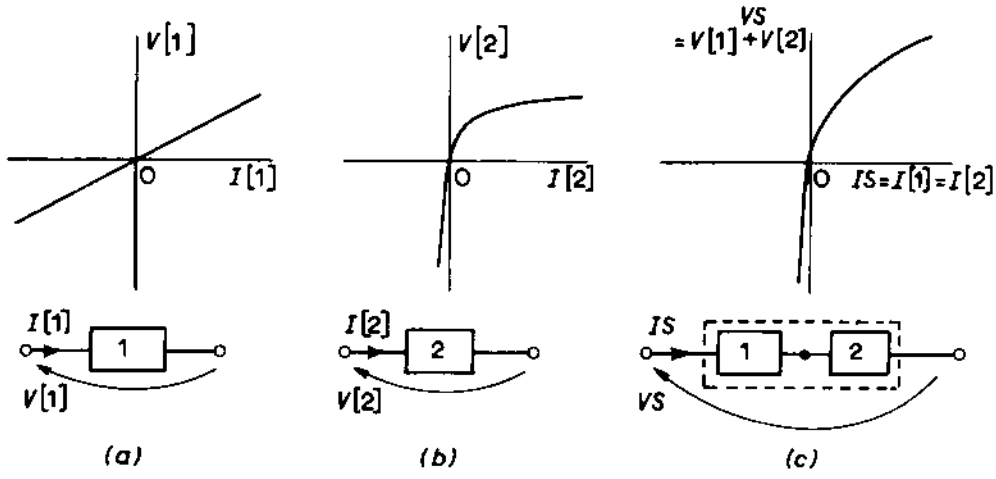


Fig. 3·7

Graphical construction to find the voltage-current characteristic of two components connected in series.

We may conclude from the construction illustrated in Fig. 3.7 that if a single component having the voltage-current characteristic shown in (c) were manufactured as a single entity, then no electrical measurement we could devise would be able to tell it apart from the series connection of components 1 and 2. The single component would be said to be **equivalent** to the series connection of components 1 and 2.

The graphical construction illustrated in Fig. 3.7 has the advantage of being valid whatever the nature of the voltage-current characteristic of each component. Nevertheless, it suffers from the disadvantage that it yields a graphical rather than an algebraic result, and would have to be repeated if, for example, the resistance of component 1 were to change.

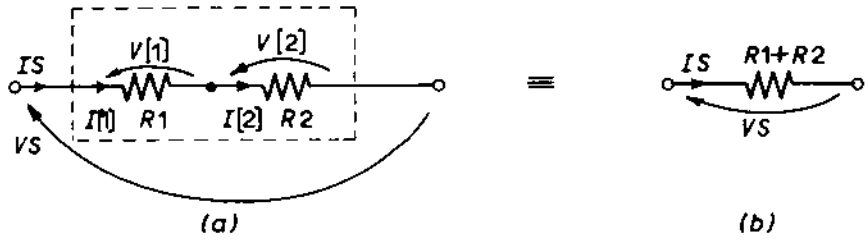


Fig. 3.8

By contrast, a simple algebraic description of the series and parallel connection of two components is possible if these components are linear and homogeneous. (Fig. 3.8). By comparison with Fig. 3.6, there is no change in the constraints

$$IS \leftarrow I[1] \leftarrow I[2]$$
$$VS \leftarrow V[1]+V[2]$$

imposed by the series connection. The difference lies in the expressions

$$V[1] \leftarrow R1 \times I[1]$$
$$V[2] \leftarrow R2 \times I[2]$$

describing the constituent components, and which permit the simple expression of VS as a function of IS:

$$VS=(R1+R2)\times IS$$

This expression tells us that the series connection of two linear homogeneous resistors of value $R1$ and $R2$ has exactly the same voltage-current characteristic as a single resistor of value $R1+R2$. Thus, the two 2-terminal components shown in parts (a) and (b) of Fig. 3.8 are said

to be equivalent, since measurements of *IS* and *VS* cannot tell them apart. We say that *R1+R2* is the equivalent resistance of the resistances *R1* and *R2* connected in series. What is important about this result is that for a certain class of components we have found that the series connection of two of them can also be regarded as a member of that class and can therefore be similarly described (i.e., by a single resistance value). The rule for obtaining a single-parameter description of the series connection of two linear resistances can usefully be embodied in a defined function:

```
     ∇ R←R1 RSS R2
[1]    R←R1+R2
     ∇
```

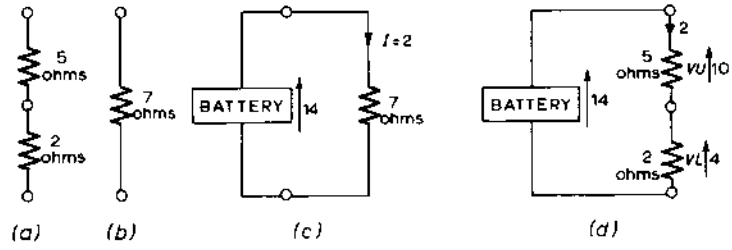


Fig. 3.9

Thus the series connection of a 5 ohm resistor and a 2 ohm resistor (Fig. 3.9)

```
      5 RSS 2
7
```

is equivalent, electrically, to a single resistance of 7 ohms. From this result it follows that if a 14-volt battery is connected across the series connection of 5 ohms and 2 ohms, the current flowing in the two resistors is

```
      □←I←(÷7)×14
2
```

as indicated in Fig. 3.9c. Reverting to the two individual resistors, the voltage across each can be calculated from Ohm's Law

```
      □←VU←5×2
10
      □←VL←2×2
4
```

This progression from separate components to their single equivalent and then back to separate components is typical of circuit analysis. The predicted voltages *VU* and *VL* satisfy Kirchhoff's voltage law:

```
      14=+/VU,VL
1
```

Note that the voltages *VU* and *VL* exhibit the same ratio as the resistance values because the current is common to both resistors:

```
      (5÷2)=VU÷VL
1
```

The rule for computing the equivalent of two linear homogeneous resistances connected in series can be employed for the series connection of more than two such resistances. For example, the series connection of the three resistances shown in Fig. 3.10 is electrically equivalent to a single resistance of 29 ohms

```
      □←R←10 RSS 17 RSS 2
29
```

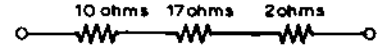


Fig. 3·10

a result which is obtained by adding the values of the resistances connected in series.

3.4 PARALLEL CONNECTION

Our discussion of the parallel connection of two 2-terminal components could be written by interchanging the pair of words 'voltage' and 'current' and the pair 'resistance' and 'conductance', etc., in the foregoing discussion of the series connection. Thus, if the same two components 1 and 2 used in Fig. 3.7 are now connected in parallel, it is convenient to employ their individual inverse description (current as a function of voltage) shown in Fig. 3.11 (a) and (b). Then, according to the constraints

```
      VP←V[1]←V[2]
      IP←I[1]+I[2]
```

imposed by the parallel connection, the relation between *IP* and *VP* (Fig. 3.11c) can be obtained by adding the values of *I*[1] and *I*[2] associated with specific values of *VP*. In this way an equivalent (nonlinear) component can be defined. Simplicity again ensues if both components are linear and

homogeneous (Fig. 3.12), since the combination of the above constraints with the description of the components

```
I[1]←G1×V[1]
I[2]←G2×V[2]
```

yields the relation

```
IP←(G1+G2)×VP
```

between the current and voltage associated with the new 2-terminal component. It is left as an exercise to define a dyadic function *GPP* whose result is the equivalent conductance of the parallel connection of two linear homogeneous conductances, the values of which are the arguments of *GPP*. The reader is also asked to investigate the manner in which *IP* divides between the two conductances *G1* and *G2* of Fig. 3.12.

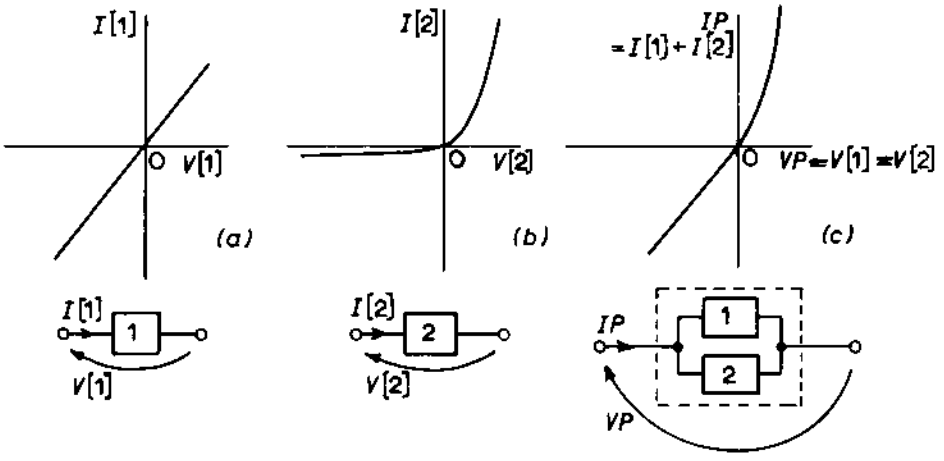


Fig.3·11

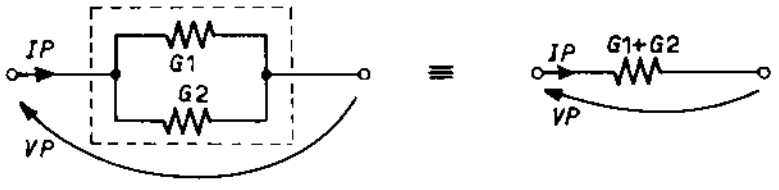


Fig. 3·12

3.5 INVERSION

In Chapter 2 it was shown that the current response I of a single linear homogeneous component to a voltage excitation V is characterized by a single parameter, the conductance G:

```
      I←G×V
```

On the other hand, the voltage response to current excitation is characterized by the inverse relation

```
      V←R×I
```

where

```
      R←÷G
```

The relation $R←÷G$ also applies to a conductance G which is the equivalent of two conductances connected in parallel. Thus, for Fig. 3.13a, the equivalent conductance G of the 1 mho and 3 mho components connected in parallel is such that their connection (Fig. 3.13b) to a 2 volt battery causes a current of 8 amps to flow:

```
      □←G←1+3
4
      G×2
8
```

If, however, a current excitation of 8 amps is applied to the parallel connection (Fig. 3.13c), the response voltage may be found by first calculating the equivalent <u>resistance</u> of the parallel connection

```
      □←R←÷G
0.25
```

and then employing the relation $V←R×I$ (Ohm's Law) to calculate the voltage:

```
      □←V←R×8
2
```

3.6 ARBITRARY CONNECTION

The value of the rules we have discovered for finding the equivalent of the series and parallel connection of two 2-terminal linear homogeneous components lies in the fact that many <u>arbitrary</u> interconnections of more than two such components can be broken down into a sequence of simple series and parallel connections. A single example will illustrate this generality.

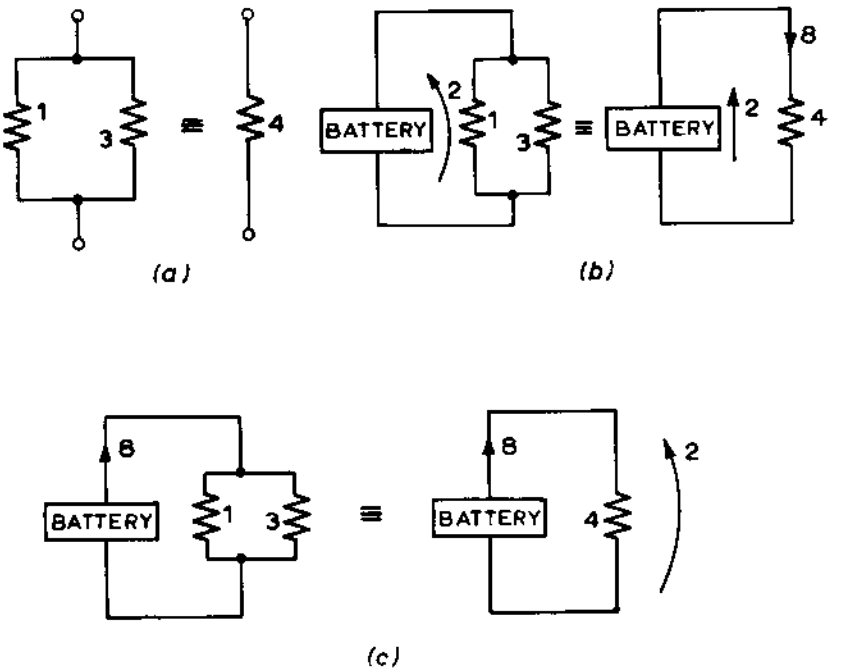


Fig. 3·13

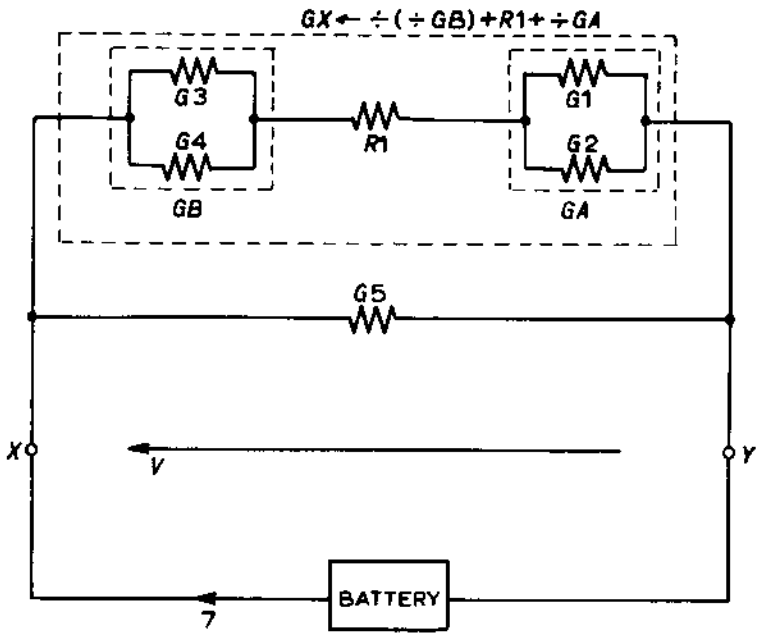


Fig. 3·14

Suppose that it is necessary to find the voltage response between terminals *X* and *Y* of the circuit of Fig. 3.14 when a current excitation of 7 amperes is applied by a suitable choice of battery. The resistance or conductance of each 2-terminal linear component is known:

G1	*G2*	*R1*	*G3*	*G4*	*G5*
2 mho	4 mho	0.5 ohm	0.1 mho	0.4 mho	1 mho

We begin by replacing the parallel connection of the components of conductance *G1* and *G2* by the equivalent conductance *GA*

```
      □←GA←G1 GPP G2
6
```

The equivalent conductance *GB* of the parallel connection of conductances *G3* and *G4* is similarly found:

```
      □←GB←G3 GPP G4
0.5
```

The series connection of three linear components whose resistances are ÷*GA*, *R3* and ÷*GB* can now be replaced by its equivalent resistance

```
      □←RX←(÷GA) RSS R1 RSS ÷GB
2.667
```

Finally, the parallel connection of one component whose resistance is *RX* (and whose conductance is therefore *GX*←÷*RX*) with another of conductance *G5* can be replaced by its equivalent conductance *G*:

```
      □←G←GX GPP G5
1.375
```

Thus, the equivalent of the entire circuit between terminals *X* and *Y* is a conductance of 1.375 mho, i.e., a resistance of 0.727 ohm.

What is interesting about this result is not that it enables us to compute the voltage response between *X* and *Y* to the excitation current mentioned in the original problem statement:

```
      R←÷G
      □←V←R×7
5.091
```

The importance of the result lies in the fact that we do not need to repeat the above determination of the equivalent

resistance between *X* and *Y* if other excitation currents must be considered:

```
      □←V←R×1 2 3 4 5 6
0.7273 1.455 2.182 2.909 3.636 4.364
```

To show that the generality of the procedure of the preceding page is a consequence of the linearity and homogeneity of the components, we now consider the circuit of Fig. 3.15 in which all components except one are linear and homogeneous: the single exception is an exponential diode whose current response to voltage excitation is described by the function *DIODEV*:

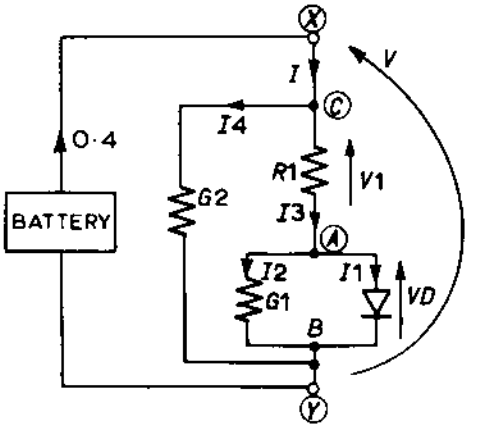


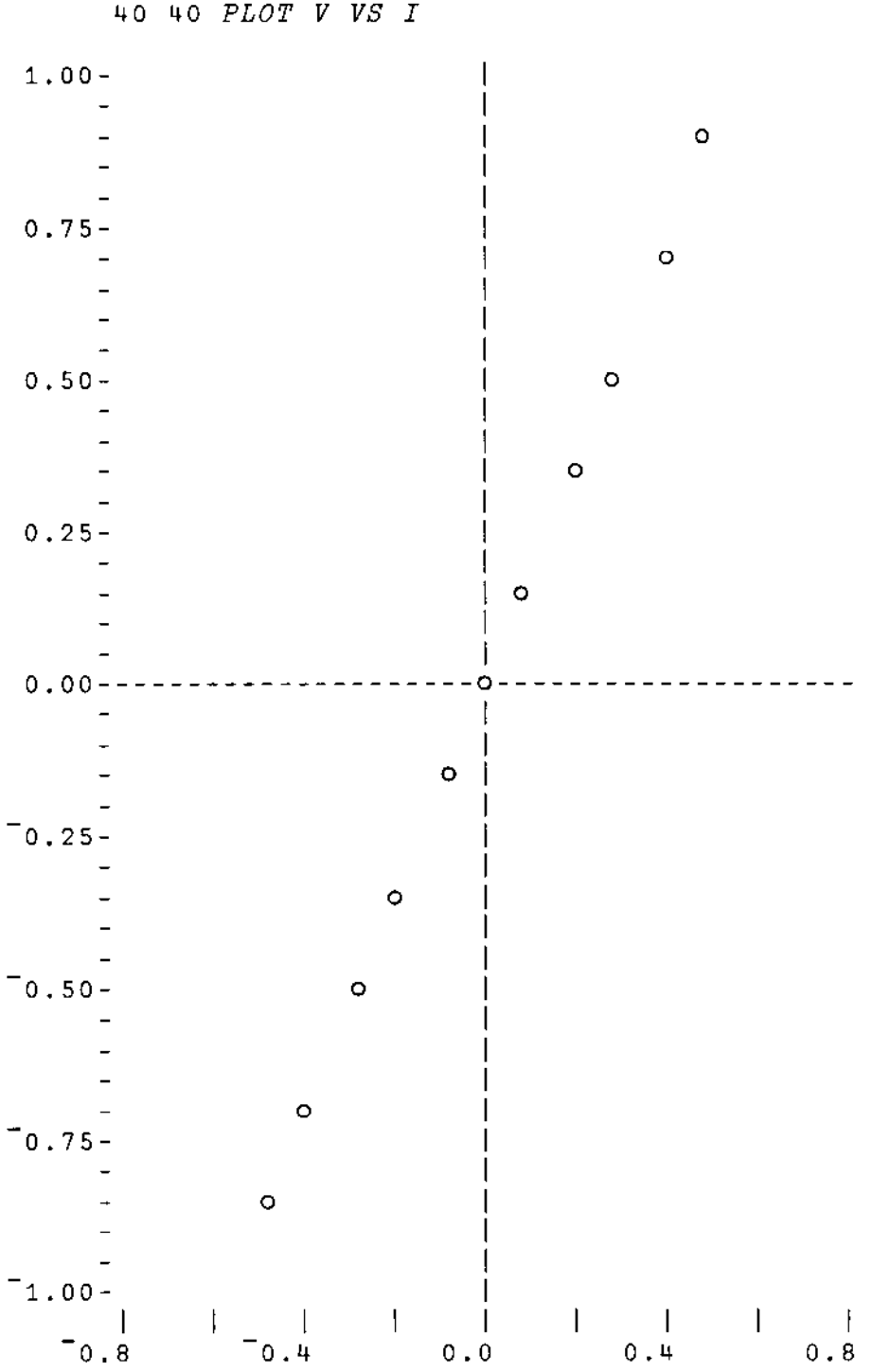
Fig. 3·15

```
    ∇ I←DIODEV V
[1]   I←IE¯11×¯1+*40×V
    ∇
```

The presence of this one nonlinear component precludes any possibility of obtaining a general description from which the response to any excitation can then simply be determined. It is necessary, here, to begin by <u>assuming</u> a particular value of the diode voltage *VD* and, by the application of Kirchhoff's Laws and the component descriptions, proceed to compute the value of *V* and *I*, the voltage and current associated with the new component having terminals *X* and *Y*. However, since we may guess the <u>wrong</u> value of *VD*, we shall guess a <u>vector</u> *VD* describing a sequence of diode voltages and determine the corresponding vectors *V* and *I*. The resulting plot of *V* versus *I* may include in its range the actual excitation current (0.4 amps) and enable us to read off the associated response voltage. The relevant calculations are annotated below.

```
                                    Relation invoked
G1←.1
R1←7
G2←.5
VD←.1×¯6+ι11
I1←DIODEV VD          Diode description
I2←G1×VD              Description of G1
I3←I1+I2              Kirchhoff's current law at A
V1←R1×I3              Description of R1
V←VD+V1               Kirchhoff's voltage law taken
                        between A, B and C
I4←G2×V               Description of G2
I←I3+I4               Kirchhoff's current law at C
```

A plot of V versus I shows that an excitation current of 0.4 amps leads to a response voltage of about 0.72 volts:



The aspects of this solution which are important to observe are the need for interpolation (to read V from the plot), the uncertainty as to whether the range of guessed values of VD encompasses the actual value, and the absence of any analytical expression for the relation between V and I.

3.6 DIODE MODELLING

In this chapter we have been presented with a number of examples in which known components are connected together and we are required to describe the new 2-terminal component that results. But the inverse problem is equally important, as an example drawn from diode modelling will show.

Knowledge of the mechanical construction and underlying physical behaviour of a semiconductor diode may lead to the suggestion of Fig. 3.16 as a model which includes two important second-order effects. One, the ohmic* resistance *RS*, is unavoidably present in the bulk of the semiconductor material. The other, an ohmic resistance *RP*, models the leakage of current around the semiconductor junction. The modelling task is now to deduce, from measurements of *V* and *I*, the values of *RS* and *RP* as well as the parameters of the internal exponential diode *D*. The techniques developed in this chapter can be used in the solution of this problem.

One approach is to establish the general form of the nonlinear relation between *I* and *V*. To do so requires a knowledge of the order of magnitude of various parameters, so we assume the following, not untypical values:

```
RS←2
RP←1E6
ID←IE¯10×¯1+*40×VD
```

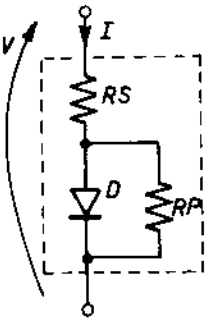


Fig. 3.16
Model of a practical semiconductor diode

Consider, first, the parallel combination of *D* and *RP*. Since they share a common voltage the graphical construction illustrated in Fig. 3.11 is appropriate. The construction (Fig. 3.17a-c) shows that the slope of the current-voltage characteristic of *D* in parallel with *RP* tends to the value *RP* for voltages which are more negative than about one-tenth of a volt. For positive voltages the slope approaches that of the characteristic of the internal diode. The subsequent connection of *RS* in series is handled by the graphical technique illustrated in Fig. 3.7, because *RS* and the remainder of the model share a common current. We see (Fig. 3.17d-f) that while the slope for negative voltages is essentially defined by *RP*, the slope approaches *RS* with increase in forward voltage. Thus we would expect the

---

* In other words, *RS* is the constant involved in Ohm's Law, relating the relevant voltages and currents.

slope of the voltage-current characteristic of the diode model to vary between approximately *RP* for *V* less than about .1 volts, via a range of values appropriate to the internal exponential diode for small positive voltages and then, for higher positive voltages, approximately *RS*.

The above discussion has established the form of the voltage-current characteristic of a practical diode. Since the determination of the characteristics of both linear and exponential functions has been fully discussed in an earlier chapter, the task of modelling a diode is left as an exercise for the reader.

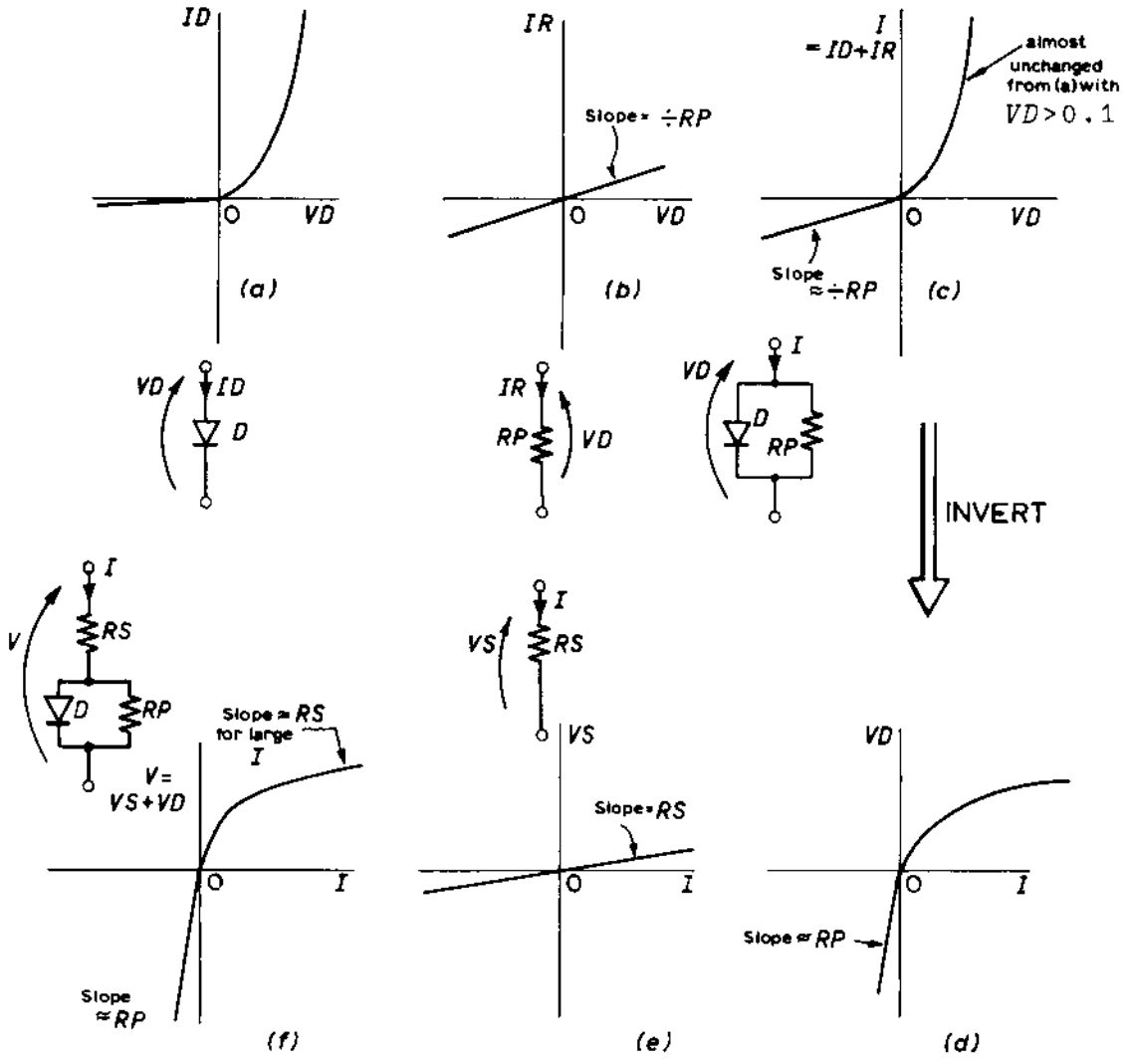


Fig.3·17 Construction to find the form of the voltage-current characteristic of the diode model of Fig.3·16. The figure should be read clockwise.

CHAPTER 4

SOURCES AND POWER

In Chapter 2 it was implied that the battery providing voltage excitation in some of the component measurements was ideal in the sense that its voltage was independent of the current it supplied. Such an ideal battery would be described by the voltage-current characteristic of Fig. 4.1, and is represented by the symbol used in the same figure. Usually it is called an ideal voltage source or, simply, a voltage source.

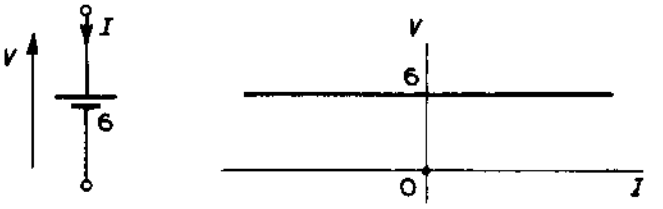


Fig. 4·1
An ideal constant voltage source

While an ideal voltage source can be simulated on the computer, as in Chapter 2, a practical battery will always depart, however slightly, from this ideal. As an increasing current is drawn from the battery by means of a variable resistance (Fig. 4.2), the battery voltage decreases, usually linearly with the current drawn.

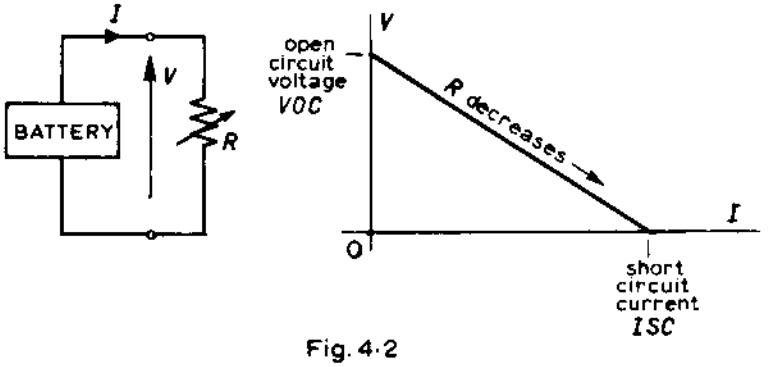


Fig. 4·2

Clearly, the measurement of any two points is sufficient to define this linear characteristic; the pair of points commonly used is the pair of intersections of the line with the axes. One intersection, described by the open- circuit voltage *VOC*, is the voltage between the battery terminals when the current is zero (i.e., when *R* is equal to infinity, or the battery terminals are left disconnected). The other intersection is defined by the short-circuit current *ISC*, which is the current supplied by the battery when *R* is zero (i.e., when the battery terminals are connected together by a wire of zero resistance). We should note, however, that a perfect measurement of *VOC* and *ISC* is precluded by the finite and non-zero resistance exhibited, respectively, by practical voltmeters and ammeters. If, as is often the case, the current supplied by a practical battery is a small fraction of *ISC*, then for most purposes it could be modelled by an ideal voltage source.

The voltage-current characteristic of Fig. 4.2 is of a form that has previously not been encountered; it is neither linear and homogeneous, nor of constant value. We are therefore faced with two choices. One is to admit an entirely new type of component, and devise some systematic way of computing its effect when connected to other components such as nonlinear diodes, resistances and other batteries. The other is to model the battery in terms of familiar components, thereby restricting the number of different types of component that we shall have to deal with. The latter approach, which we shall adopt, appears the more attractive, and is consistent with the idea of two essentially different types of component; those which supply electrical energy and those which dissipate it.

4.1 THEVENIN SOURCE REPRESENTATION

The linear decrease in battery voltage with increase in the current supplied (Fig. 4.2) strongly suggests the modelling of the battery by the series connection of an ideal voltage source and a linear resistance (Fig. 4.3). From Kirchhoff's laws and Ohm's law we may express the battery voltage *V* as a function of the current *I* supplied

$$V \leftarrow VOC - RIN \times I \qquad (4.1)$$

a function which is shown in Fig. 4.3. The name *VOC* is assigned to the ideal voltage source within the model because this is the open-circuit voltage; when the terminals are not connected externally the current *I*, and hence the voltage across the resistance, is zero, and $V \leftarrow VOC$. The value of *RIN*, the internal resistance of the battery, can be

found either by measuring the short-circuit current *ISC* and the open-circuit voltage *VOC* and then invoking Ohm's law

$$VOC \leftarrow ISC \times RIN \qquad (4.2)$$

or, equivalently, by observing from equation (4.1) that the slope of the plot of Fig. 4.3 is *-RIN*. Thus, if the values of *VOC* and *RIN* are chosen so that the voltage-current characteristics of Fig. 4.2 and Fig. 4.3 are identical, the circuit of Fig. 4.3 is an acceptable model of the battery. It is often referred to as the **Thevenin source representation** of the battery or, more concisely, the **Thevenin model**.

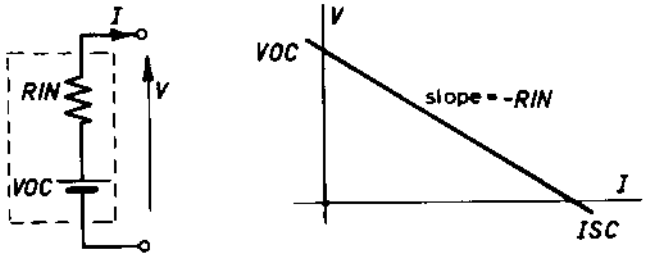


Fig. 4·3
The Thevenin Source Representation of a battery.

Given an actual battery we must determine the values of *VOC* and *RIN* if we wish to construct its Thevenin model. We have already seen that *VOC* is measured by observing the battery voltage when the battery is left disconnected. The resistance *RIN* can be deduced from the slope of the voltage-current characteristic, which in turn is determined by means of two measurements.

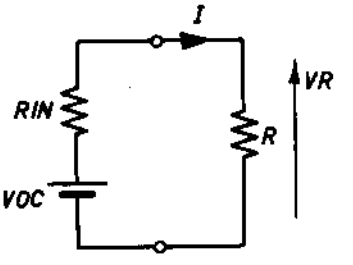


Fig. 4·4

There is an alternative method of measuring *RIN*, which is of very general interest and utility. Suppose a variable resistance is connected between the battery terminals, a situation modelled by Fig. 4.4. The equivalent resistance connected to the ideal voltage source is *RIN+R*,

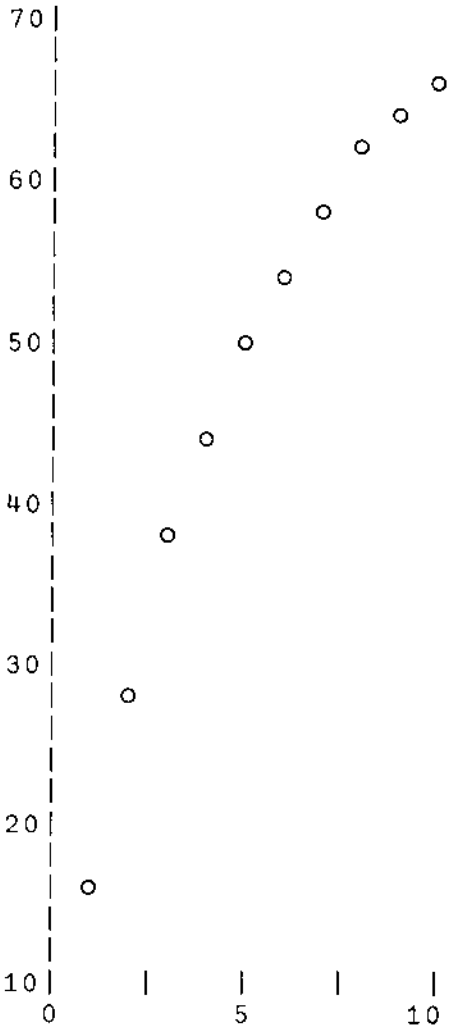
so the current *I* is equal to *VOC÷RIN+R*. From Ohm's law, the voltage *VR* across *R* is *R×VOC÷RIN+R*. Let us assume values for *VOC* and *RIN*

```
VOC←100
RIN←5
```

and investigate the variation of *VR* for a range of values of *R*:

```
R←⍳10
VR←R×VOC÷RIN+R

30 30 PLOT VR VS R
```



We see that the voltage *VR* is equal to one half of *VOC* when *R* is equal to the value of *RIN*, the internal resistance. That this will always be the case, and provide the basis of an alternative measurement of *RIN*, is clear from examination of equation (4.1).

4.2 NORTON SOURCE REPRESENTATION

Consideration of the voltage-current characteristic of a typical battery led to the proposal of a hypothetical component called an ideal voltage source and its use, in conjunction with a linear resistance, to model a practical battery. It is also possible - and useful - to propose a hypothetical ideal current source; its characteristic and symbol are shown in Fig. 4.5.

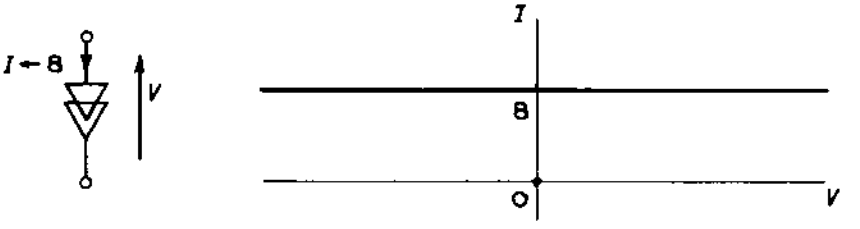


Fig. 4·5

The voltage-current characteristic, and symbol, of an ideal current source. The number associated with the symbol is the numerical value of the current (in amps) supplied by the source in the reference direction implied by the arrowheads that form the symbol.

Though we shall exploit the idea of an ideal current source extensively in this book it is convenient, here, to illustrate its application to the modelling of a battery. It is not difficult to show that the parallel connection of a constant current source and a conductance *GIN* possesses the characteristic shown in Fig. 4.6. Thus, with suitable choice of the parameters *ISC* and *GIN*, the circuit of Fig. 4.6 can model the battery, whereupon the model of Fig. 4.6 is then referred to as the Norton source representation or, simply, the Norton model, of the battery. Modelling involves the determination of *ISC* and *GIN* from measurements on the battery: the details are left as an exercise. The relation between the parameters of the two models also forms the subject of an exercise.

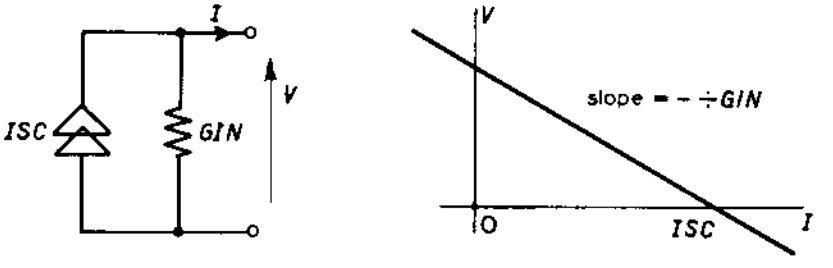


Fig 4·6

The Norton source representation of a battery.

4.3 POWER

From prior studies we know that the product of the voltage and current of a 2-terminal component is the rate at which electrical energy is supplied to the component and converted to heat, provided the reference directions for voltage and current are as shown in Fig. 2.1, repeated here as Fig. 4.7. Thus, with reference to the latter figure, the electrical energy E supplied to the component in T seconds is

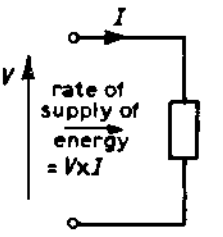


Fig. 4.7

$$E \leftarrow V \times I \times T$$

This energy, for the components being studied, is converted to heat. If V, I and T are measured in volts, amps and seconds, the energy supplied is measured in joules: 4.185 joules are required to raise the temperature of 1 gram of water by one degree Centigrade. The rate at which energy is supplied is called the power, and denoted by P. Thus, again with reference to Fig. 4.7, we say that the power P supplied to the component is

$$P \leftarrow V \times I \tag{4.3}$$

If V and I are measured in volts and amps respectively, then P is measured in watts. In this last equation only electrical quantities are involved, and time is introduced only if the total energy supplied is of interest. For the example of Fig. 4.8, the current I of 3 amps passing through a linear resistance of 5 ohms results in a voltage V of 15 volts:

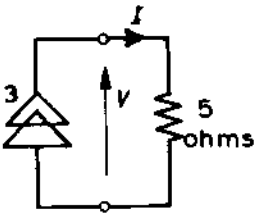


Fig. 4.8

```
      I←3
      R←5
      □←V←R×I
15
```

Keeping in mind the reference directions adopted for V and I, the power P supplied to the component is 45 watts:

```
      □←P←V×I
45
```

If the current I is allowed to pass for a time T of 10 seconds, the total energy supplied to the component and converted to heat is 450 joules:

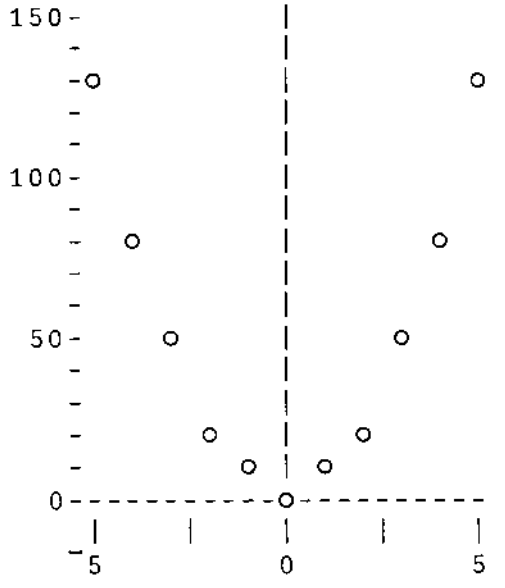
```
      T←10
      □←E←V×I×T
450
```

To explore some properties of the power associated with typical components we consider the same linear resistance R of 5 ohms to be subjected to a series of current excitations

```
      I←¯6+⍳11
```

in which current flow is both positive and negative. The voltage response vector $V←R×I$ and the vector of dissipated power $P←V×I$ are easily computed. A plot of P versus I emphasizes the fact that P is always positive; that is, power is always being supplied to the resistance irrespective of the direction of current flow:

```
      20 20 PLOT P VS I
```



The plot also illustrates the quadratic relation between power and current that is clear from examination of the alternative expression of P in terms of R and I:

```
      P←V×I
      V←R×I
      P←R×I*2
```

The equivalence of three commonly employed expressions for power:

```
      P1←V×I                P2←R×I*2              P3←(V*2)÷R
```

can easily be checked:

```
      ∧/(P1,P2,P3)=P2,P3,P1
1
```

Power is associated with any component. For example, the nonlinear diode modelled by the function *DIODEV* in Section 2.6 (and shown in Fig. 4.9) will, when subjected to the voltage excitations `V←0.2×¯6+⍳11`, dissipate a power which is the product of the diode voltage and current:

```
      ⎕←P←V×I←DIODEV V
1E¯11  8E¯12  6E¯12  4E¯12  2E¯12  0  5.96E¯9
      3.55E¯5  0.159  632  2.35E6
```

Since the elements of P span a wide range we shall not attempt to plot P. It is sufficient to draw attention to the fact that `0≤⌊/P`, i.e., the dissipated power is never negative, a property we have already encountered with the linear homogeneous component.

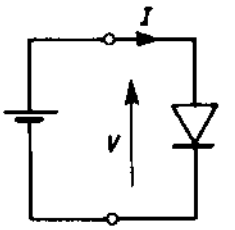


Fig. 4.9

4.4 ACTIVITY AND PASSIVITY

There are, nevertheless, 2-terminal components for which the product of V and I can be negative, so that power is capable of being supplied by the component to an externally connected circuit. The component modelled by the function *BOXV1* is an example. We arbitrarily select a range of voltage excitation (Fig. 4.10)

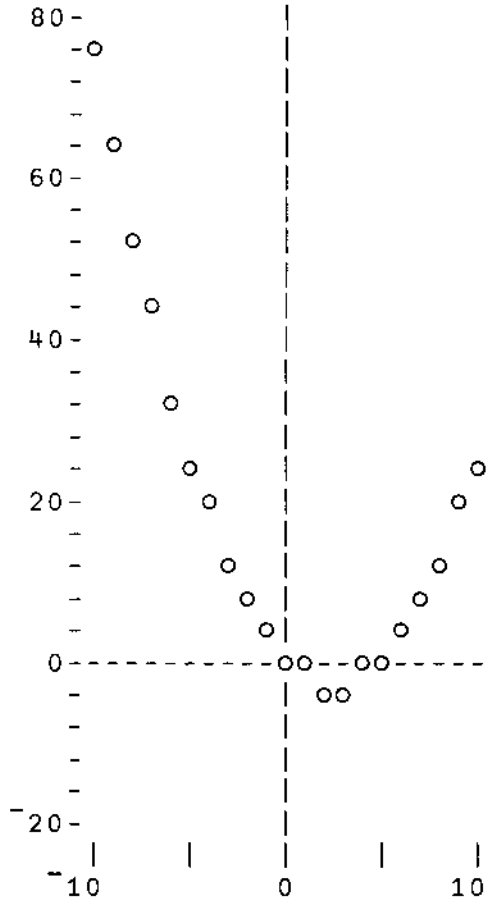
```
      V←¯11+⍳21
```

and apply it to the component to generate the corresponding vector of current response:

```
      I←BOXV1 V
```

Examination of a plot of the power `P←V×I` supplied to the component, versus the voltage V shows that, over a certain range of V, P is negative:

```
      30 30 PLOT P VS V
```



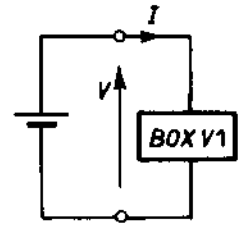


Fig. 4·10

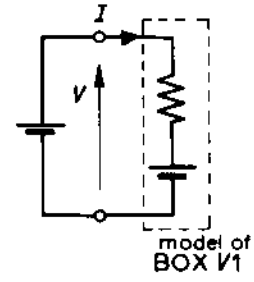


Fig. 4·11

In other words, power is actually supplied by the component to the excitation voltage source. This is not surprising if it is known that *BOXV1* is the model (Fig. 4.11) of a battery. However, the example does establish the fact that the power supplied to a 2-terminal component need not always be positive. It should also be noted, by examining the vector

```
          I
-7.5 -7 -6.5 -6 -5.5 -5 -4.5 -4 -3.5 -3 -2.5 -2 -1.5 -1 -0.5
      0 0.5 1 1.5 2 2.5
```

that current can flow **into** the positive terminal of an ideal voltage source.

Components for which the product $V \times I$ is **never negative** are called **passive** components, since they can never supply power, only convert it to heat. If a component **can** supply power - that is, if the product $V \times I$ is negative over a range of V and I - then it is called an **active** component.

4.5 MAXIMUM POWER TRANSFER

The internal resistance *RIN* of a battery tells us something useful about the way in which power can be drawn from a battery. For the example of Fig. 4.5, where

```
VOC←100
RIN←5
```

let us compute, for a selection of values of `R←⍳10` not only the current `I←VOC÷RIN+R` but also the power `P←R×I*2` converted to heat by the resistance *R*. A plot of *P* versus *R* (below, left) is seen to exhibit a maximum:

```
      30 30 PLOT P VS R
500|          o o
   |        o     o
   |      o         o
   |                  o
450|                    o
   |
   |    o
400|
   |
   |
   |
350|
   |
   |
   |
300|
   |  o
   |
   |
250|    |     |     |     |
   0          5          10
```

```
      30 30 PLOT DP VS R
250000| o
      |   o
      |     o
      |       o
      |         o
0     |           o
      |             o
      |               o
¯250000|                 o
      |
      |                   o
¯500000|
      |                     o
      |
      |
¯750000|     |     |     |     o
       0           5          10
```

In fact, by substitution for *I* in the expression for *P*, and differentiation with respect to *R*, we see that the slope, *DP*, of the plot of *P* versus *R* can be written

```
DP←(VOC*2)×((RIN+R)*2)×1+¯2×R÷RIN+R
```

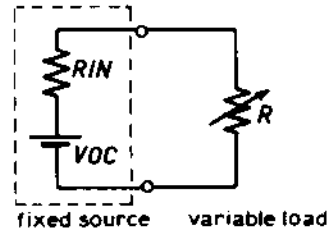


Fig. 4.12

A plot of *DP* versus *R* (above, right) confirms that when the slope *DP* is zero, and *P* is a maximum, *R* is equal to *RIN*. A second differentiation of *P* with respect to *R* does, in fact, confirm that *P* is a maximum when *R=RIN*. This result establishes the important principle of maximum power transfer:

> For the circuit of Fig. 4.12, where *VOC* and *RIN* are fixed and the resistance *R* is variable, the power dissipated in *R* is a maximum when *R=RIN*.

It is in this respect that the internal resistance of a source is of more than passing interest, since in many practical situations it is economically desirable to obtain the maximum possible power from a source.

4.6 IDEAL SOURCES AND THEIR INTERCONNECTION

It is interesting to examine various special examples of ideal sources and the restrictions upon their interconnection.

If an ideal voltage source has a value of 6 volts (Fig. 4.13a), then the voltage between its terminals is 6 volts irrespective of what is connected externally between them, and irrespective of the current flowing in the source. The situation in which an ideal voltage source has the value zero (Fig. 4.13b), and where the voltage between *A* and *B* is zero independently of what is connected externally between *A* and *B*, is a special case called a short circuit, a name used informally earlier in this chapter. Since the voltage is always zero, the source can also be represented by a zero-valued resistance because, according to Ohm's law

$$V \leftarrow R \times I$$

any current will then give rise to zero voltage. A short circuit is represented symbolically by a single line, as shown in Fig. 4.13b. Representation of a short circuit by a line is consistent with our previous use of lines to represent connections between components, since we implicitly assume no voltage difference between the two ends of a connection.

One way in which two ideal voltage sources can be connected together is shown in Fig. 4.13c. This diagram tells us that the voltage between *A* and *B* is 7 volts, irrespective of the current in the source, and that the voltage between *B* and *C* is 3 volts, again irrespective of

the current in the source. From Kirchhoff's voltage law we conclude that the voltage between A and C is always 10 volts, so that this series connection of two ideal voltage sources can be represented by a single, equivalent, ideal voltage source of 10 volts (Fig. 4.13c).

The parallel connection of two ideal voltage sources of different value (Fig. 4.13d) is forbidden. One symbol signifies that the voltage between A and B is 7 volts, the other that it is 3 volts. Since these assertions are contradictory, such a connection is forbidden. The single exception is, of course, when the two ideal sources have the same value. As a particular example of a forbidden connection (Fig. 4.13e), an ideal voltage source of non-zero value cannot be terminated by a short-circuit. The connection of two actual batteries, each represented by a Thevenin equivalent (Fig. 4.13f), is permitted. It is easy to calculate, for this circuit, currents and voltages which do not violate Kirchhoff's laws and which are not inconsistent with the use of symbols describing ideal voltage sources.

Similar rules and restrictions apply to the interconnection of ideal current sources, and some form the subject of exercises. A special example of an ideal current source is one whose value is zero (Fig. 4.13g). Since this implies that the current flowing in the source is zero

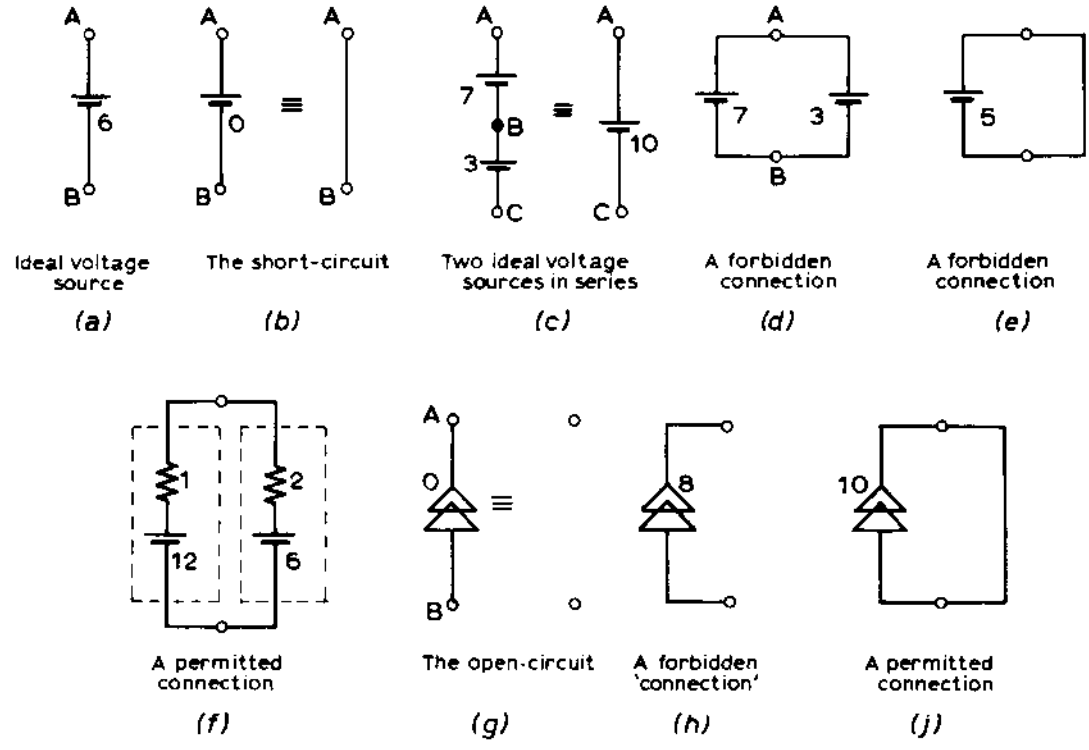


Fig. 4·13

irrespective of the voltage across it, the source can be replaced by an open circuit - that is, by the absence of any connection - between *A* and *B*, as shown in Fig. 4.13g. It follows, therefore, that the series connection of an ideal current source of non-zero value and an open circuit (Fig. 4.13h) is forbidden, since the current *I* which is common to both sources cannot at the same time be both zero and non-zero. What is permitted is the connection of an ideal current source of non-zero value to a short circuit (Fig. 4.13j): no incompatibilities arise, and we are able to determine the current and voltage associated with each component.

4.7 LOAD-LINE CONSTRUCTION

Graphical methods for determining the voltage-current characteristic of the series and parallel connection of two 2-terminal components were presented in Chapter 3. Since no restriction was placed on the nature of these components we would expect the graphical methods to work for the case when one of the components is a non-ideal source. Let us explore this idea.

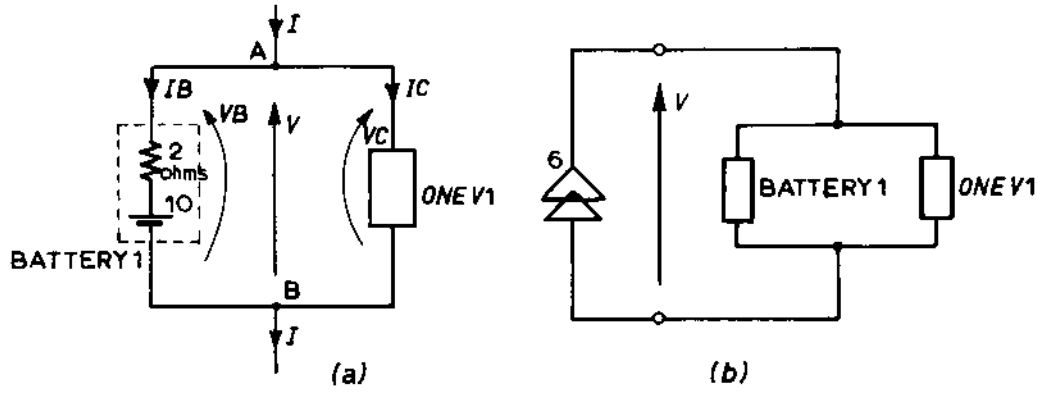


Fig. 4.14

Fig. 4.14a shows the model of a battery, represented by its Thevenin equivalent, connected to a nonlinear component. The two components are described, respectively, by the functions *BATTERY1* and *ONEV1*

```
    ∇ I←BATTERY1 V
[1]   I←(V-10)÷2
    ∇
```

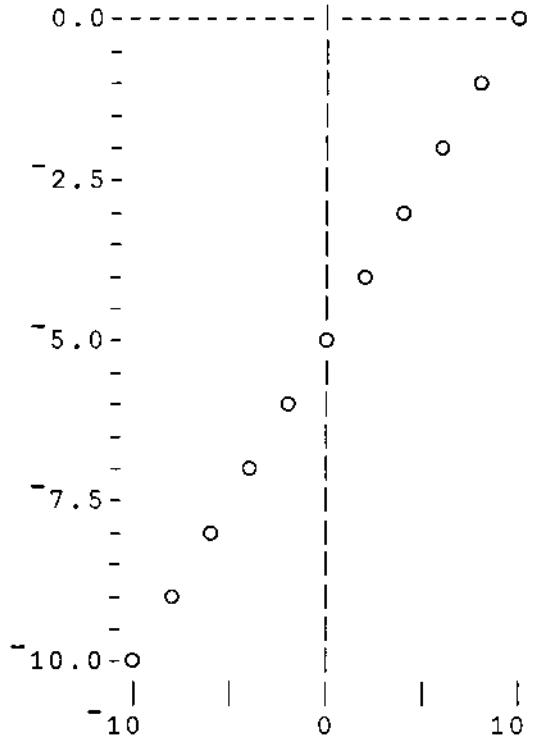
```
    ∇ I←ONEV1 V
[1]   I←(.5×V)+5E¯3×V*3
    ∇
```

Their separate voltage-current characteristics for a selected vector of voltage excitation are shown below.

```
      □←V←¯12+2×⍳11
¯10  ¯8  ¯6  ¯4  ¯2   0   2   4   6   8  10
```

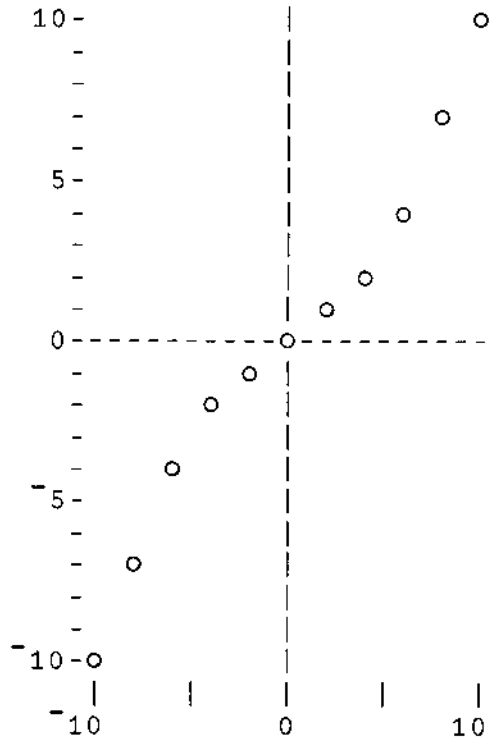
```
IB←BATTERY1 V

20 20 PLOT IB VS V
```



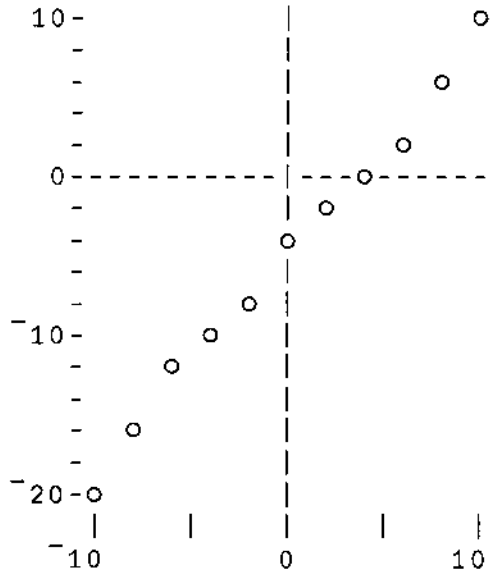

```
IC←ONEV1 V

20 20 PLOT IC VS V
```



Since the two components can be said to be connected in parallel (they share a common voltage) the graphical method illustrated in Fig. 3.11 can be employed to determine the voltage-current characteristic of the complete circuit connected between terminals *A* and *B* in Fig. 4.14a. The procedure is illustrated in Fig. 4.15; for the present example the plot corresponding to part (c) of Fig. 4.15 is

```
20 20 PLOT (I←IB+IC) VS V
```



Thus if an excitation current of 6 amps is now connected between *A* and *B*, as in Fig. 4.14b, the resulting value of *V* can be read from the plot of Fig. 4.15c: it is 8 volts.

But the situation shown in Fig. 4.14b is not often of interest. What is of interest, however, is a method of analysing the simple circuit of Fig. 4.14a, when *I* is zero. If *I* is zero, the corresponding value of *V* (4.5 volts) can be noted from the plot of Fig. 4.15c and allows the values of *IB* and *IC* to be found by reference to parts (a) and (b) of Fig. 4.15. This appears to be a new use for the graphical methods introduced in Chapter 3: that of analysing a simple circuit consisting of a Thevenin source connected to a component which may be nonlinear.

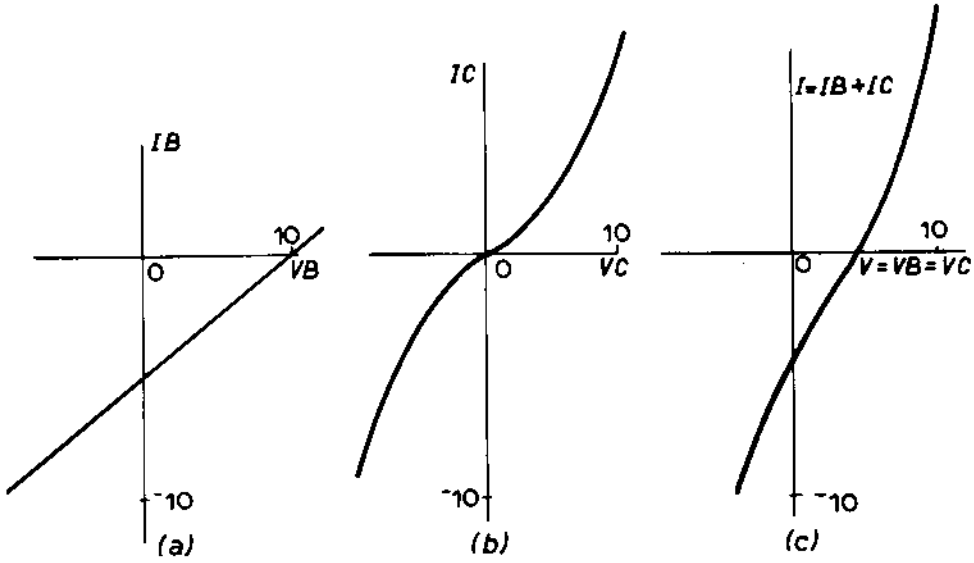


Fig. 4·15

Graphical construction to find the relation between I and V in the circuit of Fig. 14·14 a.

It is valuable to explore this example a little further. We used the graphical method first described in Fig. 3.11 because the two components of Fig. 4.14a can be regarded as being connected in <u>parallel</u>. But this figure could equally well be regarded as a series connection. What is important, however, is not what we <u>call</u> the connection but the <u>constraints</u> it imposes. If Fig. 4.14a is redrawn as Fig. 4.16, we see that <u>two</u> constraints are imposed: Kirchhoff's voltage law requires that

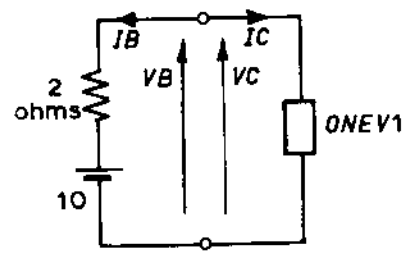


Fig. 4·16

$$VB = VC$$

while Kirchhoff's current law imposes the constraint

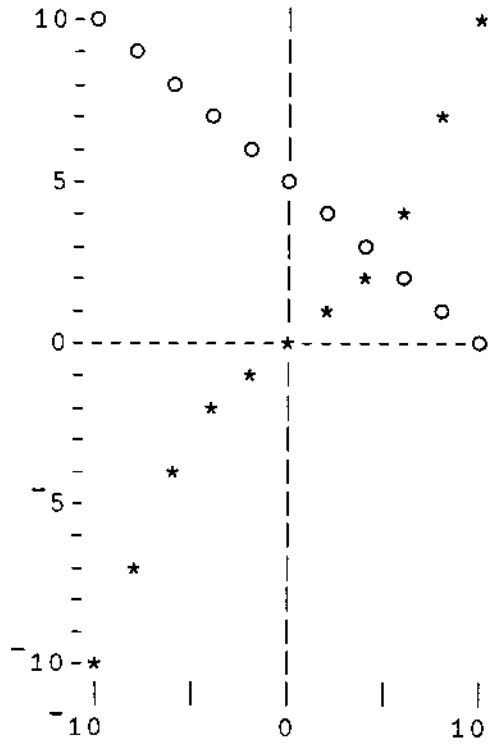
$$(-IB)=IC$$

We may therefore plot the relation between $-IB$ and VB, and that between IC and VC, on a common graph (Fig. 4.17).

The point defining the values of VB and IB must be on the curve of $-IB$, and the point defining the values of VC and IC must be on the curve IC. Since $VB=VC$ and $(-IB)=IC$, it follows that the intersection of the two plots with VB and VC equal to 4.5 and $-IB$ and IC equal to 2.75 describes the condition of the circuit of Fig. 4.16.

We have already computed the numerical data required to plot the curves of Fig. 4.17:

```
20 20 PLOT ((-IB) VS V) WITH IC VS V
```



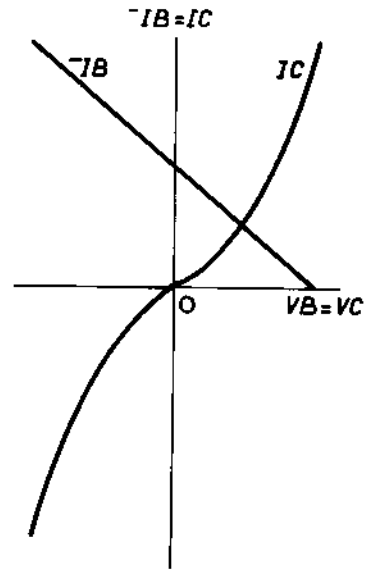


Fig.4.17

The graphical construction illustrated in Fig. 4.17 is valid for any 2-terminal components, and is commonly referred to as the load-line construction; in particular, the plot of the Thevenin source characteristic (the curve of $-IB$ in Fig. 4.17) is referred to as the load-line.

CHAPTER 5

SIGNALS

The purpose of many electrical systems is to transmit information from one place to another - for example, from a teletypewriter to a computer or from one telephone user to another. The voltages and currents we have studied in the previous chapters have been constant in value, and are represented graphically by a plot whose value remains constant with time, as in Fig. 5.1a. But such a waveform cannot transmit information: to transmit information some variation with respect to time is necessary. The timevarying voltage waveform of Fig. 5.1b, for example, might describe the voltage of a microphone when exposed to speech. Thus, to determine the behaviour of circuits intended for the transmission of information, it is necessary to know how to predict the response of a circuit to an excitation which varies with time.

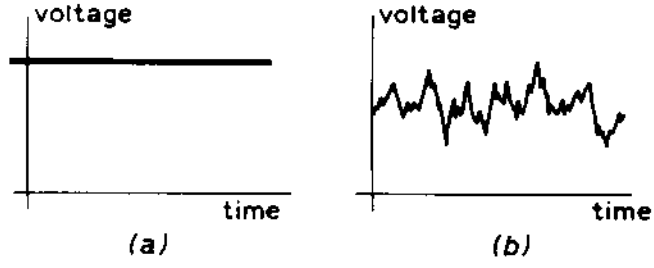


Fig. 5.1

5.1 CONSTANT AND TIMEVARYING COMPONENTS

In the most general case the source of an information-bearing waveform will generate, across its two terminals, a timevarying voltage whose average value is non-zero. An example of such a waveform is shown in Fig. 5.2a, where a square waveform has been chosen for simplicity. Since our interest lies principally in the variation with time of the circuit's response it seems

appropriate, and will later be seen to be justified, to resolve the voltage into the sum of two components, a constant voltage (Fig. 5.2b) and (Fig. 5.2c) a timevarying voltage whose average value is zero. If the waveform of Fig. 5.2a is that of an ideal voltage source, then the waveforms of Fig. 5.2b-c also describe ideal sources, and are represented by the symbols shown in the figure. Since the voltage components are added to obtain the original waveform, it follows from Kirchhoff's voltage law that the ideal voltage source whose waveform is shown in Fig. 5.2a can be modelled by the series connection of the voltage sources of Fig 5.2b and c. An alternative symbol for a timevarying voltage source whose average is not zero is also shown in part a. If we are concerned only with the information associated with a voltage or current waveform, then the important aspect of the waveform is its variation with time; absolute values are unimportant. We shall therefore tend to focus attention upon the zero-average, timevarying component of a waveform, frequently referred to as the signal component, or simply the signal.

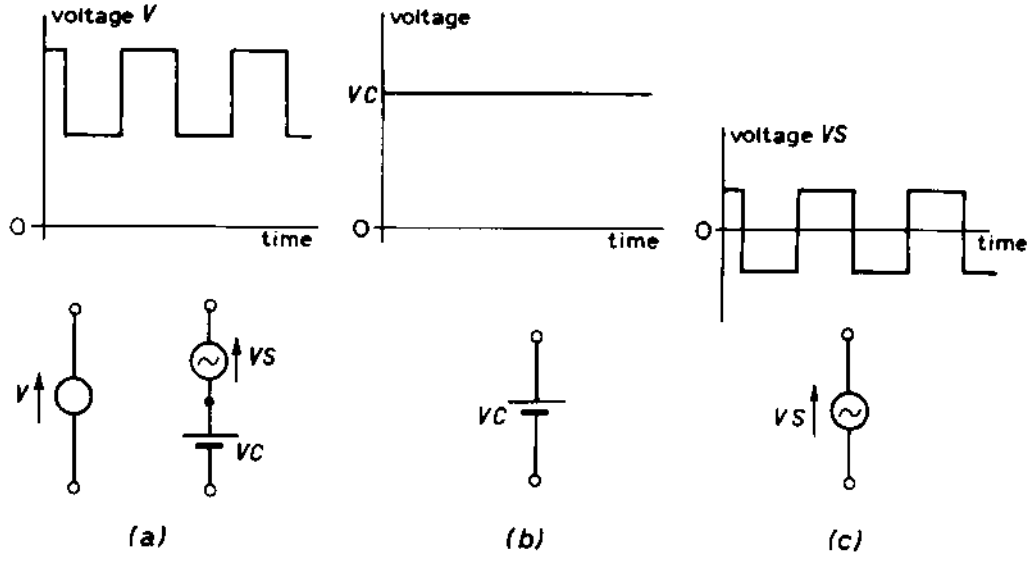


Fig. 5·2

The resolution of a timevarying voltage into constant and zero-average components.

If the waveform of Fig. 5.2a is that of the open-circuit voltage of a practical source, then as an increasing current is drawn the absolute values of the voltage will decrease while preserving the waveform (Fig. 5.3). In this case the practical source is represented by a Thevenin or Norton equivalent possessing some internal resistance (Fig. 5.4).

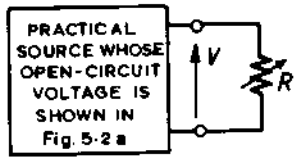


Fig. 5·3

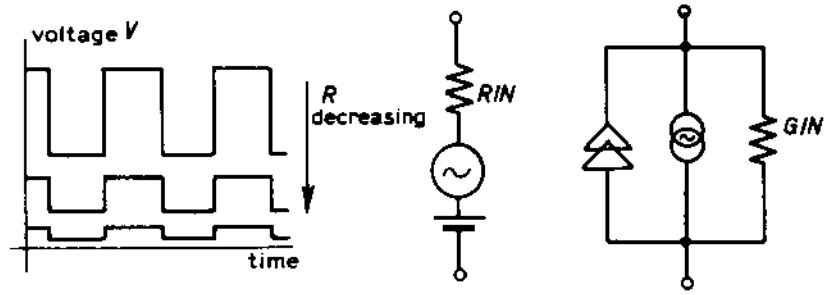


Fig. 5·4
The Thevenin and Norton equivalents of a timevarying source.

5.2 SIGNAL RESPONSE OF A LINEAR COMPONENT

Fig. 5.5 shows the circuit model of an ideal timevarying voltage source connected to a linear homogeneous component whose conductance is 4 mho. The problem is to find the waveform of the component current I. In the method we shall adopt first this is all we shall find. Then it will be shown how a more general approach can yield more insight and power.

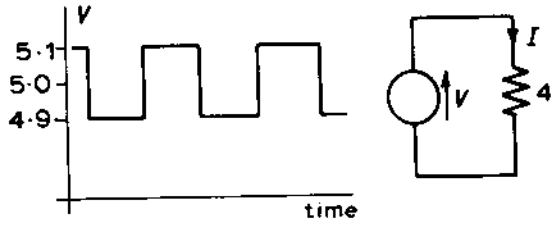


Fig. 5·5
Problem : to find the waveform of I

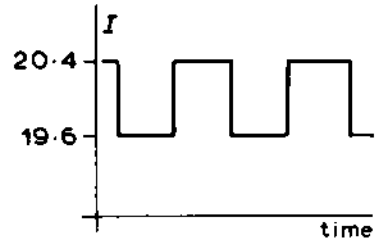


Fig. 5·6
Current response of the circuit of Fig. 5·5

In view of the waveform it is necessary only to invoke Ohm's law to compute the two currents corresponding to the two values of voltage, from knowledge of the component conductance:

```
      G←4
      □←IMAX←G×5.1
20.4
      □←IMIN←G×4.9
19.6
```

The current waveform* is therefore as shown in Fig. 5.6.

---

* In assuming a perfect square waveform for the response current we are ignoring energy storage effects (due to capacitance and inductance) which are always present.

Far more insight is obtained if, as shown in Fig. 5.7, the voltage excitation of the 4 mho conductance is represented as the sum of a constant component VC and a varying component ΔV whose average value is zero. In other words, ΔV is the signal component of the excitation voltage. For the problem of Fig. 5.5 ΔV assumes only two values (0.1 and ¯0.1); for more typical signals such as a speech waveform ΔV is a continuous function of time, but could be approximated by a sequence of discrete values.

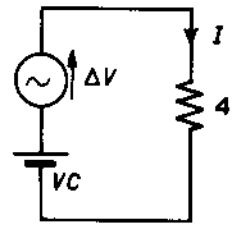


Fig. 5-7
Pertaining to the problem of Fig. 5-5

Let us explore the current response I for a range of values of both the average (VC) and signal (ΔV) components of the voltage excitation

```
VC←⍳5
ΔV←0.1×¯3+⍳5
```

by forming an array AV of voltage excitation in which all possible combinations of VC and ΔV are systematically arranged:

```
⎕←AV←VC∘.+ΔV
 .8   .9  1.0  1.1  1.2
1.8  1.9  2.0  2.1  2.2
2.8  2.9  3.0  3.1  3.2
3.8  3.9  4.0  4.1  4.2
4.8  4.9  5.0  5.1  5.2
```

The corresponding array AI of current response is computed by reference to Ohm's law:

```
⎕←AI←G×AV
 3.2  3.6  4.0  4.4  4.8
 7.2  7.6  8.0  8.4  8.8
11.2 11.6 12.0 12.4 12.8
15.2 15.6 16.0 16.4 16.8
19.2 19.6 20.0 20.4 20.8
```

When the signal voltage ΔV is set to zero, the current response is described by the array AIC:

```
⎕←AIC←G×VC∘.+(⍴ΔV)⍴0
 4  4  4  4  4
 8  8  8  8  8
12 12 12 12 12
16 16 16 16 16
20 20 20 20 20
```

The **difference** between the current response arrays *AI* and *AIC* is, therefore, due entirely to the presence of the signal voltage ΔV.

```
      □←AΔI←AI-AIC
 ¯.8    ¯.4    .0    .4    .8
 ¯.8    ¯.4    .0    .4    .8
 ¯.8    ¯.4    .0    .4    .8
 ¯.8    ¯.4    .0    .4    .8
 ¯.8    ¯.4    .0    .4    .8
```

Since ΔV describes the timevarying nature of the voltage excitation, the array $A\Delta I$ describes the timevarying nature of the response. Let us now inspect this array to see if we can discover any general properties of the current response.

Any one row of $A\Delta I$ is associated with a fixed value of *VC* and a vector of equally spaced values of ΔV. Thus, the first row of $A\Delta I$

```
      □←ΔI←AΔI[1;]
¯0.8 ¯0.4 0 0.4 0.8
```

describes the variation ΔI in current around the value associated with a value of *VC* of 1 volt. The calculation

```
      (DIFF ΔI)÷DIFF ΔV
4 4 4 4
```

shows that the relation between variations in current and voltage is linear and homogeneous; i.e.,

$$\Delta I \leftarrow G \times \Delta V \qquad (5.1)$$

where $G \leftarrow 4$. Since for any change in voltage excitation there is a proportionate change in current response, the average value of ΔI is zero and ΔI can be called the signal component of the response current. Clearly, ΔI and ΔV have the same shape of waveform. The constant *G* is called the **signal conductance**. Because the rows of $A\Delta I$ are identical, it follows that equation (5.1) holds independently of the value of *VC*. There is therefore no need to know the constant component of the excitation when computing the signal component of the response.

Similarly, there is no need to know the signal component ΔV of the excitation when finding the constant component *IC* of the response because, as a consequence of equation (5.1), the average value *IC* of the response current

is unchanged by the presence of a signal. Any column of *AIC* describes the vector of constant components of *I*

```
      □←IC←AIC[;1]
4 8 12 16 20
```

associated with the vector *VC* of constant components of *V*. Element-by-element division

```
      (DIFF IC)÷DIFF VC
4 4 4 4
```

shows that the relation between *IC* and *VC* is linear and homogeneous, and characterized by the same conductance as for the signal components.

```
      IC←4×VC                                          (5.2)
```

The important conclusion to be drawn from the above discussion is that the response to the constant and signal components of the excitation can be determined independently (from equations 5.2 and 5.1 respectively), and the results added to find the complete response.* Thus, the problem of Fig. 5.5 can be solved by resolving the excitation voltage into constant and signal components, computing the associated current responses, and then adding them to find the total current response:

```
      VC←5
      ΔV←0.1 ¯0.1
      IC←G×VC
      ΔI←G×ΔV
      □←I←IC+ΔI
20.4 19.6
```

If desired, the calculation of *IC* and *ΔI* can be associated, respectively, with the circuits of Fig. 5.8a and b: Fig. 5.8b, which embodies equation (5.1), is called the signal equivalent circuit.

The conclusion described above could equally well have been drawn from the relation between the excitation and response of Fig. 5.7 (that is, *IC+ΔI* is equivalent to *G×(VC+ΔV)*), in view of the fact that the operation of

---

* This result is known as the Principle of Superposition, and will be encountered again in Chapter 10.

multiplication distributes over that of addition (that is, $G\times(VC+\Delta V)$ is equivalent to $(G\times VC)+G\times\Delta V$).

$$(IC+\Delta I)\leftarrow(G\times VC)+G\times\Delta V$$

The use of matrices, though unnecessary in this case, was introduced as a preliminary to the following discussion concerning the signal response of a nonlinear component.

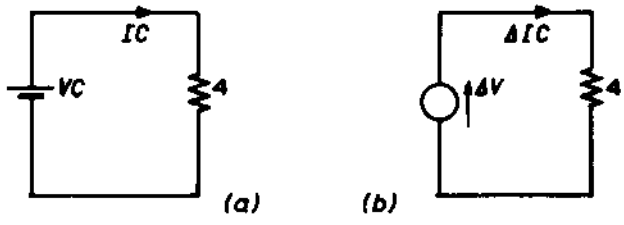


Fig. 5·8
Circuits for calculating the (a) constant and (b) signal component of the response of the circuit of Fig. 5·7

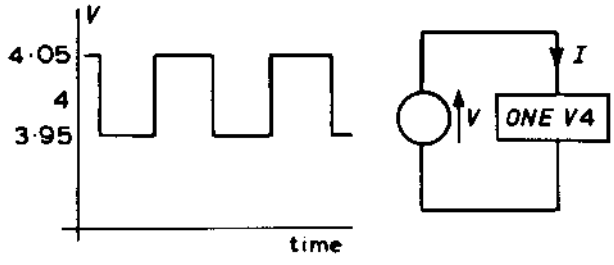


Fig. 5·9
Problem : find the waveform of I

5.3 SIGNAL RESPONSE OF A NONLINEAR COMPONENT

Fig. 5.9 shows the model of a circuit consisting of a timevarying voltage source connected to a nonlinear 2-terminal component *ONEV4*. It is required to find the waveform of the component current.

We again adopt a general approach, and explore the current response of the component (Fig. 5.10) by means of the same array of voltage excitation

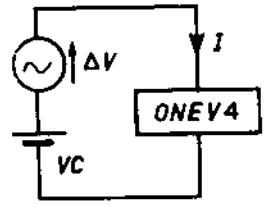


Fig. 5·10

AV

.8	.9	1.0	1.2	1.2
1.8	1.9	2.0	2.2	2.2
2.8	2.9	3.0	3.2	3.2
3.8	3.9	4.0	4.2	4.2
4.8	4.9	5.0	5.2	5.2

in which the constant and signal components of the excitation are systematically arranged. The associated array of current response

AI←ONEV4 AV

will, in general, differ from the current response array *AIC* due to the constant voltage alone:

```
      AIC←ONEV4 VC∘.+(ρΔV)ρ0
      □←AΔI←AI-AIC
 ¯.849    ¯.427     .000     .433     .873
¯1.020    ¯.514     .000     .526    1.060
¯1.300    ¯.661     .000     .679    1.380
¯1.710    ¯.868     .000     .892    1.810
¯2.240   ¯1.140     .000    1.170    2.360
```

The difference is due entirely to the signal voltage ΔV.

Each row of *AΔI* is associated with a particular value of the constant voltage *VC*. For *VC*=1 volt, the calculation

```
      □←ΔI←AΔI[1;]
¯0.849 ¯0.427 0 0.433 0.873

      (DIFF ΔI)÷DIFF ΔV
4.22 4.27 4.33 4.4
```

shows that the signal current response to signal voltage excitation is <u>not</u> linear and homogeneous. Nevertheless, it is approximately so if the signal amplitudes are <u>small</u>: how small they must be depends upon the use to which the circuit will be put. Thus, under this small-amplitude restriction, we may assume the linear and homogeneous relation

$$\Delta I = 4.22 \times \Delta V \qquad (5.3)$$

between signal current and voltage for the value of *VC* under consideration (1 volt), and define a signal equivalent circuit (Fig. 5.11) by means of which the signal response may be computed. However, because the circuit is valid for only <u>small</u> values of ΔV and ΔI it is called the <u>small-signal equivalent circuit</u>. Similarly, the conductance appearing in equation 5.3 can be called the <u>small-signal conductance</u>.

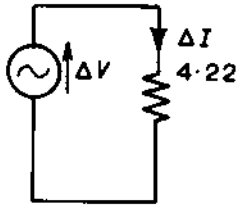


Fig. 5·11
Small-signal equivalent circuit for Fig. 5·10 when VC = 1 volt.

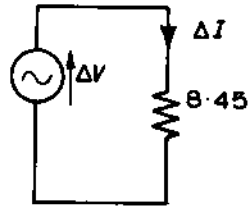


Fig. 5·12
Small-signal equivalent circuit for Fig. 5·10 when VC = 4 volts.

What happens if *VC* is now changed to 4 volts? Examination of the appropriate row of *AΔI*

```
      ΔI←AΔI[4;]
      (DIFF ΔI)÷DIFF ΔV
8.45 8.68 8.92 9.17
```

shows that small values of ΔI and ΔV may again be related, to a good approximation, by a linear, homogeneous expression,

$$\Delta I = 8.45 \times \Delta V \qquad (5.4)$$

but now the small-signal conductance (8.45 mho) differs from the value appropriate to $VC=1$ volt. Again, we may compute the signal behaviour by means of a small-signal equivalent circuit (Fig. 5.12), but we must remember that the small-signal equivalent conductance of the component is a function of *VC*.

The <u>average value</u> *IC* of the current response will be unaffected by the presence of a signal if equations 5.3 and 5.4 are valid, an approximation which is acceptable only if ΔI and ΔV are sufficiently small, i.e., under small-signal conditions. In this event, *IC* is identical with the response due to *VC* acting alone:

```
      IC←ONEV4 VC
```

What conclusions can be drawn from the above discussion? Principally, that the method employed to find the signal response of a linear component can be used for a nonlinear component, but with the following restrictions:

> The signal components of current and voltage must be small in amplitude.
>
> The small-signal conductance of the component must first be calculated from knowledge of the average value of the excitation.

Thus, for the problem stated in Fig. 5.9 we first resolve the excitation into constant and signal components:

```
      VC←4
      ΔV←0.05 ¯0.05
```

and then calculate the small-signal conductance associated with the value of *VC*:

```
      □←G←1↑(DIFF AΔI[4;])÷DIFF ΔV
¯8.45
```

The signal component of I is therefore

```
      □←ΔI←G×ΔV
¯0.422 0.422
```

and the constant component of I is

```
      □←IC←ONEV4 VC
22.4
```

The waveform of I is shown in Fig. 5.13.

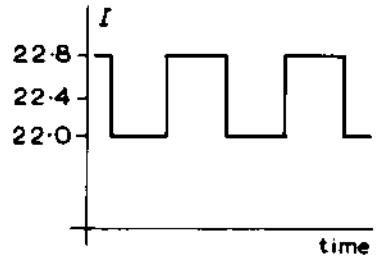


Fig. 5·13

The waveform of the current I in the circuit of Fig. 5·9.

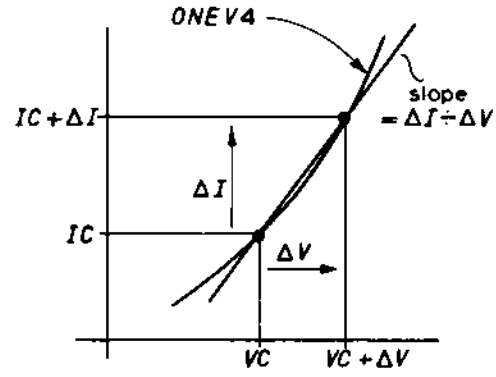


Fig. 5·14

The above discussion can be related to a graphical plot of the function *ONEV4* (Fig. 5.14). Reference to equation 5.3 shows that the small-signal conductance, used to calculate the signal component of the response, is equal to the slope of the function at the point (VC,IC) known as the operating point of the component. The slope of a function describing a component was first encountered in Chapter 2, where it was called the local conductance to emphasize its dependence upon the operating point. The plot of Fig. 5.14 also enables some impression to be gained of the way in which the actual signal response will depart from that described by the linear homogeneous relations 5.3 and 5.4.

5.4 DISTORTION

In the two examples we considered in this chapter the signal was described by a square waveform. Since the signal varied instantaneously between only two values, the response current possessed the same shape of waveform irrespective of the nature of the component.

But in many applications (e.g., speech transmission by telephone) the signal varies continuously over a range of values: what can then be said about the waveform of the signal response? To find a partial answer, we consider the component *ONEV4* again, but now subjected to an excitation (Fig. 5.15) which is the sum of a constant voltage of 2 volts and a sinusoidally varying voltage whose peak-to-peak amplitude is 3 volts. The waveform of the response current can easily be obtained via the graphical construction shown in Fig. 5.16, and exhibits two properties of interest.

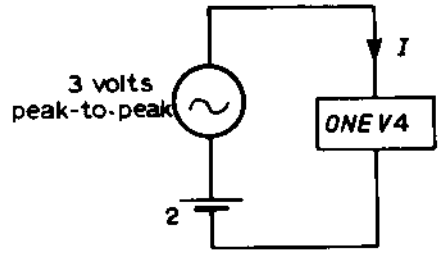


Fig. 5·15

First, the variation in response current is nonsinusoidal, so that the excitation and response do not share the same shape of waveform. In this case it is said that distortion has occurred; normally, distortion is an unwanted effect, as any high-fidelity enthusiast will appreciate! It is important to note, however, that the term distortion is used only to describe a difference in shape: a perfect amplifier having an excitation and response of vastly different magnitude but identical shape possesses no distortion.

It is also seen from Fig. 5.16 that the average value of the response current is not equal to the current (*ONEV4* 2) associated with the average value (2 volts) of the excitation voltage, an effect due entirely to the nonlinearity of the component. However, it is obvious from the figure that as the amplitude of the signal component of the excitation decreases, then so does the distortion as well as the departure of the average value of the current from the value *ONEV4* 2.

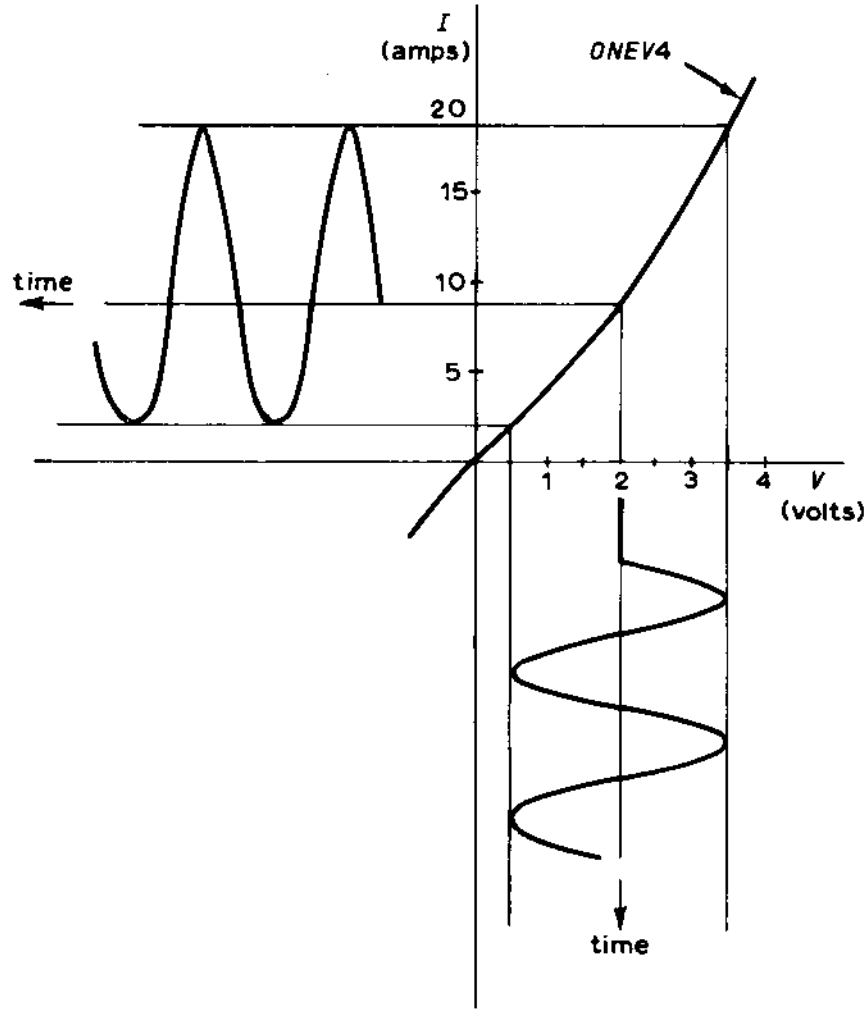


Fig. 5·16
Illustrating the nonsinusoidal current response of the nonlinear 2-terminal component ONEV4 to sinusoidal voltage excitation.

CHAPTER 6

MEASUREMENT AND MODELLING OF 3-TERMINAL COMPONENTS

Circuits composed solely of 2-terminal components can perform many useful functions. But there are some functions - such as amplification - which they cannot conveniently perform, and which call for the use of components having more than two terminals. The most frequently employed is the 3-terminal component (Fig. 6.1), whose measurement and modelling are discussed in this chapter. The value of a 3-terminal component lies principally in the separation that it affords between the input and output of what is, in effect, a control device: a small amount of (controlling) power supplied to one pair of terminals can control the supply of a substantially larger amount of (controlled) power by another pair of terminals. The transistor, vacuum-tube and relay are examples of such three-terminal control devices.

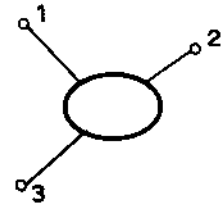


Fig. 6.1
A 3-terminal component

6.1 CHOICE OF EXCITATION AND RESPONSE

Just as for the 2-terminal component, we first determine the number of relevant voltages and currents. At first sight it would appear that three currents (one at each terminal) and three voltages (between each pair of terminals) must explicitly be taken into account. However, Kirchhoff's current law allows one current to be deduced from knowledge of the other two; also, the value of one voltage (between terminals 1 and 2, say) can be determined, by invoking Kirchhoff's voltage law, from knowledge of the other two voltages (between terminals 1 and 3, and 2 and 3). In our measurement and modelling of the 3-terminal component we shall, therefore, confine our attention to two observable currents and two observable voltages, respectively described by the 2-element vectors I and V. As for the 2-terminal component, we shall consider the interior of the component to be inaccessible to measurement.

How can the two currents and two voltages be chosen? A completely arbitrary choice is certainly permitted, but later discussions are considerably simplified if the elements of I and V are chosen in the manner shown in Fig. 6.2. Here, the two terminals associated with $I[1]$ and $I[2]$ are also the positive reference terminals for the voltages $V[1]$ and $V[2]$ respectively. Expressed another way, the common (negative) reference point for the elements of V is the terminal whose current does not appear explicitly as an element of I, and which must be deduced from I by invoking Kirchhoff's current law.

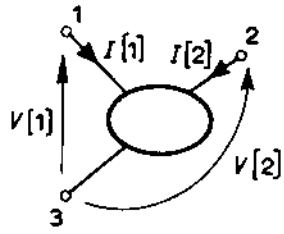


Fig. 6.2

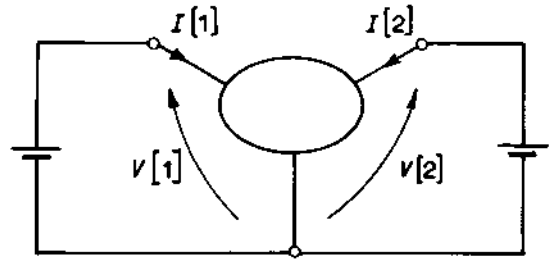


Fig. 6.3

What is the general form of the relation between the vectors I and V? To answer this question, consider the component to be subjected to voltage excitation (Fig. 6.3), so that the elements of V are constrained in value by the ideal voltage sources. The currents $I[1]$ and $I[2]$ are then a function of the two ideal voltage sources. This functional dependence is most usefully expressed in the form

$$I \leftarrow F\ V \qquad (6.1)$$

The 2-element vector V describes the voltage excitation of the component, and the 2-element vector I describes its current response.

To obtain a mathematical model of a component's voltage-current behaviour a sufficient number of measurements must be carried out. First, we shall employ voltage excitation and observe current response, a measurement for which the circuit of Fig. 6.4 is appropriate. In practice, of course, the measurements would be carried out in the laboratory, but for ease of discussion will again be simulated by means of function evaluations using the computer.

Consider the 3-terminal component described by the function *TWOV*1. A single measurement, in which the

excitation voltages $V[1]$ and $V[2]$ are arbitrarily set at 5 volts and 2 volts respectively,

```
      □←I←TWOV1 5 2
11 3
```

returns the corresponding values* of $I[1]$ and $I[2]$ (11 amps and 3 amps respectively), but provides negligible insight into the nature of the component. We might, therefore, seek to establish the effect of $V[1]$ by employing a sequence SV of voltage excitations in which $V[1]$ is varied while $V[2]$ is held constant:

```
      □←SV←(0,⍳6),[0.5] 2
0 1 2 3 4 [5] 6
2 2 2 2 2 [2] 2
```

As for the previous excitation (shown boxed), the second element of the new excitation is 2. The current response

```
      □←SI←TWOV1 SV
 ¯4  ¯1   2   5   8 [11] 14
  8   7   6   5   4 [ 3]  2
```

(in which the previous response is shown boxed) certainly provides more insight into the function, though it tells us nothing about the effect of the excitation voltage $V[2]$.

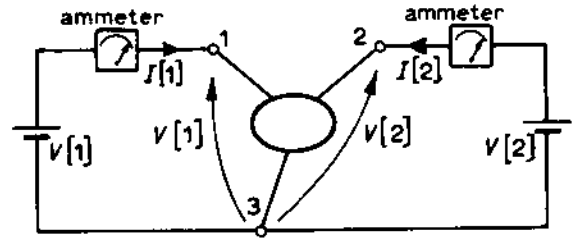


Fig. 6·4
Voltage excitation of a 3-terminal component

6.2 SYSTEMATIC MEASUREMENT

The function $TWOV1$ is most effectively explored by allowing **both** $V[1]$ **and** $V[2]$ to vary over specified ranges, but in an ordered manner which permits some pattern to be discerned in the resulting responses.

---

* The value of the current entering the component via terminal 3 is, according to Kirchhoff's current law, a simple function of I:

```
      -+/I
 ¯14
```

Since all possible combinations of $V[1]$ and $V[2]$ are involved, a suitable table in which to systematically record the measurements is that shown in Fig. 6.5. The upper matrix contains values of $I[1]$, and the lower one contains values of $I[2]$. To illustrate the nature of this table the measurements already carried out on the component $TWOV1$ are recorded in Fig. 6.5.

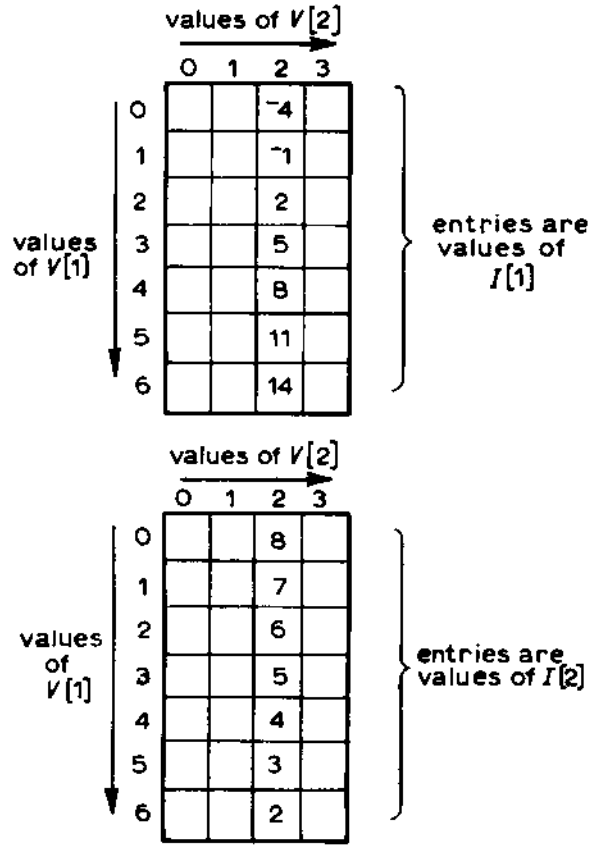


Fig. 6·5

A means of recording a systematic series of measurements on a 3-terminal component.

Let us carry out the measurements which will yield a complete table of current response for the function $TWOV1$. We choose, as successive values of the excitation voltage $V[1]$, the elements of the vector $V1$ and, as successive values of $V[2]$, the elements of $V2$:

```
V1←0 1 2 3 4 5 6          V2←0 1 2 3
```

All possible combinations of these values of $V[1]$ and $V[2]$ can be generated by applying the following function PP to the arguments $V1$ and $V2$

```
     ∇ R←A PP B
[1]    R←(A∘.+0×B),[.5](0×A)∘.+B
     ∇
       ⎕←AV←V1 PP V2
 0 0 0 0
 1 1 1 1
 2 2 2 2
 3 3 3 3
 4 4 4 4
 5 5 5 5
 6 6 6 6

 0 1 2 3
 0 1 2 3
 0 1 2 3
 0 1 2 3
 0 1 2 3
 0 1 2 3
 0 1 2 3
```

The resulting 3-dimensional array *AV* of voltage excitation is seen to contain every possible pair of excitation voltages, arranged in a manner which is consistent with the table of Fig. 6.5. The first and second planes of *AV* are associated, respectively, with the voltages *V*[1] and *V*[2], so that any pair of corresponding elements from the two planes of *AV* represents one of the possible excitations. For example, the vector *A*[;6;3]

```
       ⎕←V←AV[;6;3]
5  2
```

describes the excitation first used in our exploration of the function *TWOV*1.

Since the first dimension of *AV* is 2, it is suitable as an argument of *TWOV*1, and results in the following array *AI* of current response:

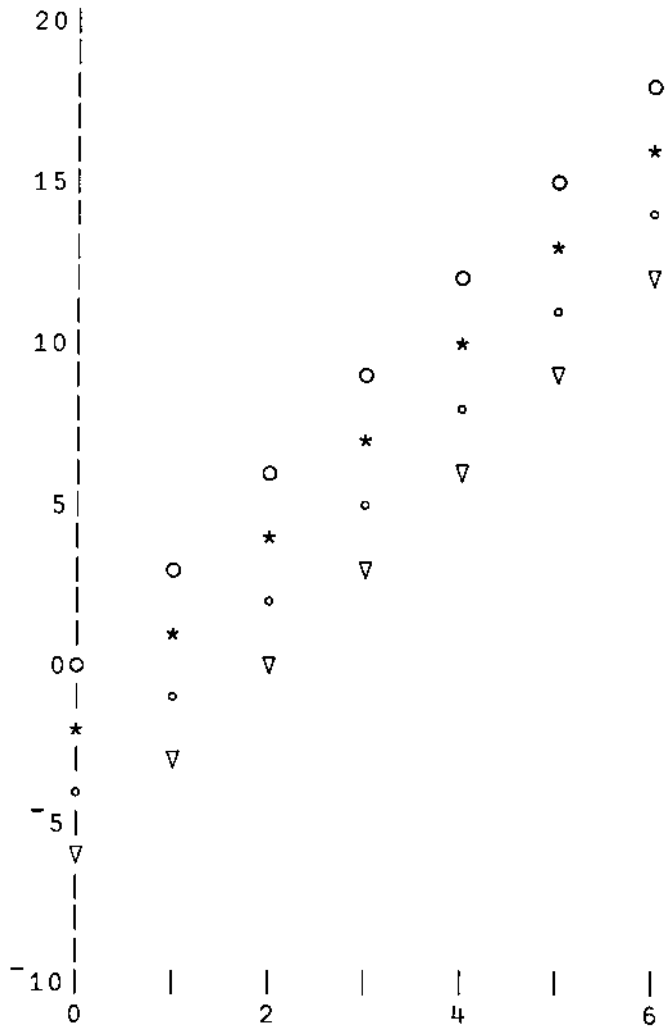
```
       ⎕←AI←TWOV1 AV
   0  ¯2  ¯4  ¯6
   3   1  ¯1  ¯3
   6   4   2   0
   9   7   5   3
  12  10   8   6
  15  13  11   9
  18  16  14  12

   0   4   8  12
  ¯1   3   7  11
  ¯2   2   6  10
  ¯3   1   5   9
  ¯4   0   4   8
  ¯5  ¯1   3   7
  ¯6  ¯2   2   6
```

The array *AI* has the same form as the entries in the table (Fig. 6.5) originally proposed for recording our measurements. The first plane, *AI*[1;;], contains values of *I*[1], and the second plane contains values of *I*[2]. The current responses associated with earlier measurements can be located within *AI*.

The two arrays, *AV* and *AI*, constitute a set of measurements upon the 3-terminal component *TWOV*1. It is not easy, however, from an examination of the excitation array *AV* and the response array *AI*, to discern the relation between excitation and response. Far more insight is provided by a graphical representation of the measurements. Let us, for example, examine a plot of *I*[1] versus *V*[1] in which the different symbols are associated with the different values of *V*[2] described by the elements of *V*2:

```
30 30 PLOT AI[1;;] VS V1
```



Two properties of this plot are immediately noticeable:

(1) the relation between *I*[1]and *V*[1] is apparently linear for any given value of *V*[2];

(2) the curves are equally spaced.

On the basis of this evidence we shall assume that the current *I*[1] is linearly related to both *V*[1] and *V*[2]. The test

```
      □←I←TWOBOXV1 0 0
0 0
```

shows that the relation is also homogeneous. We therefore propose, with some measure of confidence, that the relation between *I*[1] and the two voltages is of the form

$$I[1]\leftarrow(G11\times V[1])+(G12\times V[2]) \qquad (6.2)$$

where *G*11 and *G*12 are constants. The symbol *G* is used because the constants have the same dimension as conductance; the numbers occurring in the name *G*12 signify that this conductance refers to the portion of *I*[1] due to *V*[2].

6.3 THE LINEAR HOMOGENEOUS 3-TERMINAL COMPONENT

The linearity of the relation between *I*[1], *V*[1] and *V*[2] can be verified, and the constants *G*11 and *G*12 determined, by use of the function *DIFF*.

For an arbitrarily selected value (the third) of *V*[2], the calculation

```
      (DIFF AI[1;;3])÷DIFF V1
3 3 3 3 3 3
```

confirms the linear dependence of *I*[1] upon *V*[1] over the range of voltage explored, and shows that the value of *G*11 in equation (6.4) is 3 mho. This should be clear from examination of equation (6.2): when *V*[2] is maintained constant, an increase of 1 volt in *V*[1] leads to an increase of *G*11 amps in *I*[1].

In the same way the linear dependence of *I*[1] upon *V*[2] can be determined, and the value of the constant *G*12 found; here the value of *V*[1] is chosen randomly from the seven employed in the measurements:

```
      (DIFF AI[1;?7;])÷DIFF V2
¯2 ¯2 ¯2
```

The value of *G*12 is ¯2 mho.

A similar investigation of the dependence of the other current, *I*[2], upon the same pair of voltages leads to another linear homogeneous relation:

```
I[2]←(G21×V[1])+(G22×V[2])
```
(6.3)

in which the values of *G21* and *G22* are found to be ¯1 mho and 4 mho respectively.

By means of measurements using the function *TWOV1* it has been established that the two response currents *I*[1] and *I*[2] are linearly and homogeneously related to the excitation voltages *V*[1] and *V*[2] by the expressions

```
I[1]←(G11×V[1])+(G12×V[2])
I[2]←(G21×V[1])+(G22×V[2])
```
(6.4)

But *I*[1] and *I*[2] are the elements of *I*, the vector of response currents, and *V*[1] and *V*[2] are the elements of *V*, the vector of excitation voltages. A more compact form of the relation (6.4) between response and excitation is therefore

```
I←G+.×V
```
(6.5)

where

```
      □←G←2 2⍴3 ¯2 ¯1 4
 3  ¯2
¯1   4
```

For confirmation, we see that the voltage excitation

```
      V←5 2
```

and its associated current response

```
      □←I←G+.×V
11 3
```

are in agreement with an earlier measurement. Just as for the 2-terminal component, it is useful to define a model of the component *TWOV1*

```
    ∇ I←MODEL V
[1]   I←(2 2⍴3 ¯2 ¯1 4)+.×V
    ∇
```

and to compare calculations using this model with previous measurements

```
      ∧/,(MODEL AV)=TWOV1 AV
1
```

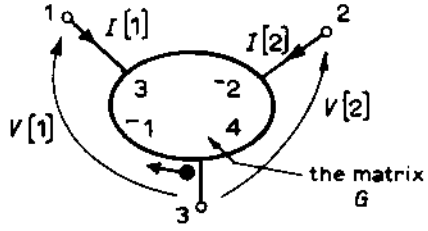
The model is seen to be satisfactory.

Ohm's law for a linear, homogeneous 2-terminal component and the equation (6.5) describing a linear, homogeneous 3-terminal component are similar in structure:

2-terminal component	3-terminal component
`I←G×V`	`I←G+.×V`

In fact, the latter relation can be regarded as a generalization of Ohm's law to components with more than two terminals. In the former case scalar multiplication is involved in the computation of response, in the latter an inner product. For the linear two-terminal component one parameter, the conductance, serves to describe its voltage-current behaviour, whereas four parameters - the four elements of G in equation (6.7) - are required to describe a linear 3-terminal component. We refer to equations (6.4) and (6.5) as the conductance description of the 3-terminal component, and to G as the conductance matrix. The elements of G, namely $G[1;1]$, $G[1;2]$, $G[2;1]$ and $G[2;2]$, are known as the conductance parameters of the component.

Four conductance parameters constitute a complete description of a 3-terminal component only if the voltage reference terminal has been identified. In this book, I use the convention of placing a dot against the terminal selected as the voltage reference terminal (Fig. 6.6) and associating, with this dot, an arrow which indicates the order in which the terminals are numbered. Often, the conductance matrix is inscribed within the component symbol, as in Fig. 6.6.




```
I ← I[1],I[2]
V ← V[1],V[2]
I ← G+.×V
```

Fig. 6.6

The reader may justifiably ask, at this point, whether a linear, homogeneous 3-terminal component is an acceptable model of any practical component. The answer is two-fold. First, for certain regions (e.g., positive $V[1]$ and $V[2]$) of the voltage-current characteristics of a practical component, a linear homogeneous relation may be an acceptable model in that it can be used to predict circuit behaviour to a reasonable accuracy. But of far greater importance is the fact that a linear, homogeneous expression can accurately relate changes in voltage and current about an operating point, and is employed very extensively in the prediction of the small-signal behaviour of circuits such as transistor amplifiers.

6.4 MEASUREMENT OF THE CONDUCTANCE MATRIX

If a 3-terminal component is known to be described by a relation of the form of equation (6.5), the determination of the elements of the matrix *G* need not involve the extensive series of measurements employed in the previous section; these additionally served the purpose of establishing linearity and homogeneity.

Examination of equation (6.5) shows that if *V*[1] is set to one volt and *V*[2] is zero, then the corresponding currents

```
      □←I←TWOV1 1 0
3 ¯1
```

are identical, numerically, with the values of *G*[1;1] and *G*[2;1]. Similarly,

```
      □←I←TWOV1 0 1
¯2 4
```

yields the numerical values of *G*[1;2] and *G*[2;2]. It is convenient, in fact, to incorporate both these excitation vectors into a single excitation matrix

```
      □←V←2 2ρ1 0 0 1
1 0
0 1
```

(the unit matrix of dimension 2) since the resulting current response matrix

```
      □←I←TWOV1 V
 3  ¯2
¯1   4
```

is then the required conductance matrix. But this result

```
      G←TWOV1 UNIT 2
```

(where *UNIT* 2 is the unit matrix of dimension 2) should not cause surprise: examination of the function (*MODEL*) which models the component *TWOV1* shows that application of the function *TWOV1* to a matrix *V* is equivalent to the matrix multiplication of *V* by the conductance matrix *G*. If *V* is a 2 by 2 unit matrix, the result will, of course, be *G*.

These simple calculations correspond to the physical measurements shown in Fig. 6.7. Setting *V*[2] to zero in the calculation of *G*[1;1] and *G*[2;1] corresponds to the replacement of the excitation voltage *V*[2] by a short-circuit (Fig. 6.7a) in the measurement of these parameters. If *V*[1] is chosen to be one volt, as shown, measurement of *I*[1] and *I*[2] yields the values of *G*[1;1] and

$G[2;1]$ directly. In a similar manner the values of $G[1;2]$ and $G[2;2]$ may be measured (Fig. 6.7b). In view of the nature of this method of measuring the conductance parameters they are often referred to as the short circuit parameters, and G as the short-circuit matrix description of the linear, homogeneous 3-terminal component.

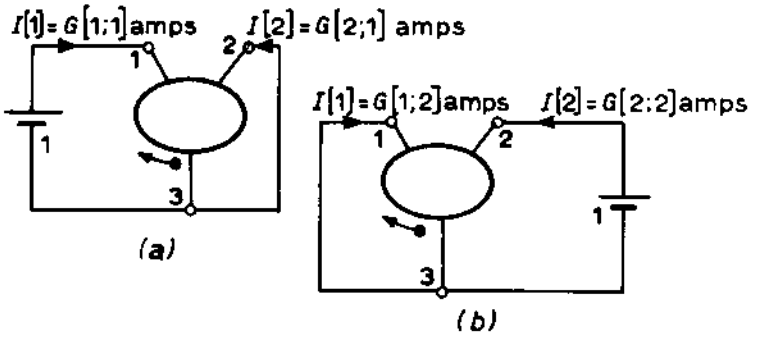


Fig.6·7

6.5 THE INVERSE FUNCTION

We have made measurements upon the component *TWOV1*, modelled it by the linear homogeneous function

```
I←G+.×V                                                    (6.5)
```

and determined the elements of the 2 by 2 conductance matrix G. Equation (6.5) is, in effect, a procedure for determining the current response to any voltage excitation. But now suppose that we are asked to find the voltage response to current excitation (Fig. 6.8). To do so we must define the procedure

```
V←R+.×I                                                    (6.6)
```

where we shall call R the resistance matrix of the component. Since the function of equation (6.6) is to undo that of equation (6.5), these functions are inverses. In fact, if we apply the two functions sequentially

```
I←G+.×R+.×I
```

to generate the current vector from itself, we immediately see that $G+.\times R$ must be the unit matrix of dimension 2,2. We thus have a relation between the matrices G and R which is similar to that between the conductance and resistance of a linear 2-terminal component

2-terminal component	3-terminal component
$G\times R$ 1	$G+.\times R$ 1 0 0 1

(Note that if G and R are 1 by 1 matrices, $G+.\times R$ is the same as $G\times R$). In this way, even though we do not yet know the value of R, we know that no further measurements need be carried out to find R: there is a direct relation between G and R.

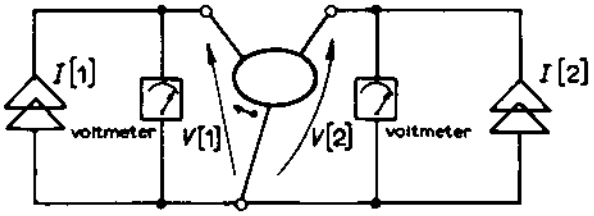


Fig.6·8
Current excitation of a 3-terminal component

For the 2-terminal component, the function $G\times V$ produced a scalar current I, and the function $R\times I$ then produced the original voltage V. G and R were said to be inverse, and the procedure for computing G from R, and vice versa, was

```
G←÷R
R←÷G
```

In a similar manner, for the 3-terminal component, the function $G+.\times V$ (6.5) produces a vector I, and the function $R+.\times I$ (6.6) then produces the original vector V. Again, therefore, we may say that G and R are mutually inverse. However, since they are matrices, we say that G is the **matrix inverse** of R, and vice versa. The symbol ⌹ denotes the function of matrix inversion, and once more, we see a similarity between the expressions associated with 2- and 3-terminal components:

2-terminal component	3-terminal component
$G\leftarrow\div R$ $R\leftarrow\div G$	$G\leftarrow⌹R$ $R\leftarrow⌹G$

To illustrate these matters for the case of the previous example:

```
      G
 3 ¯2
 ¯1  4
      ⎕←I←G+.×V←5 2
11 3
      ⎕←R←⌹G
0.4   0.2
0.1   0.3
      R+.×I
5 2
```

We see that the original voltage vector is produced. The diagrammatic representation of these calculations is shown in Fig. 6.9.

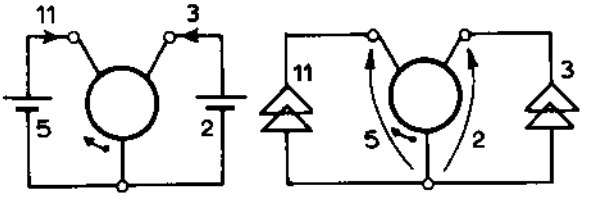


Fig. 6.9

In the above discussion we formed the matrix product of R (the inverse of G) and I to generate V:

```
V←(⌹G)+.×I
```

An alternative notation is

```
V←I⌹G
```
(6.7)

and clearly resembles the relation $V←I÷G$ associated with a 2-terminal linear component. In the same way we can write, from (6.6)

```
I←V⌹R
```
(6.8)

Relations (6.7) and (6.8) may be considered to be a generalization of Ohm's law to 3-terminal components, and are similar in structure to the corresponding relations for a 2-terminal component:

2-terminal component	3-terminal component
`V←I÷G` `I←V÷R`	`V←I⌹G` `I←V⌹R`

To discover yet another similarity between the two components we recall how the conductance matrix could be generated directly from the relation

```
G←TWOV1 V←UNIT 2
```

which is equivalent to

```
G←G+.×UNIT 2
```

```
∇ U←UNIT N
[1] U←(⍳N)∘.=⍳N
∇
```

This is only a special case of a more general relation in which the values selected for the excitation voltages need not be 0 and 1. To see why this is so, we refer to equation (6.5):

```
I←G+.×V                                                    (6.5)
```

If V is a 2 by 2 matrix, post-multiplication of the right-hand side by the inverse of V (that is, $G+.\times V+.\times\boxminus V$) yields the matrix G, since the inner product of V with its inverse is a 2 by 2 identity matrix. Post-multiplication of the left-hand side yields $I+.\times\boxminus V$. We have therefore established a relation

```
G←I+.×⊟V                                                   (6.9)
```

which allows the conductance matrix to be computed directly, but which does not specify the value of V. For an arbitrary voltage excitation matrix V and the corresponding current response I:

```
      □←V←2 2⍴4?100
 46 54
 22  5

      □←I←TWOV1 V
 94  152
 42  ¯34
```

the relation (6.9) yields the expected conductance matrix:

```
      □←G←I+.×⊟V
 ¯3  ¯2
 ¯1   4
```

A similar calculation

```
      □←R←V+.×⊟I                                           (6.10)
 0.4       0.2
 0.1       0.3
```

produces the resistance matrix. Relations (6.9) and (6.10) recall similar expressions which are relevant to the measurement of the conductance and resistance of a 2-terminal component:

2-terminal component	3-terminal component
`G←I×÷V` `R←V×÷I`	`G←I+.×⊟V` `R←V+.×⊟I`

The resistance matrix has here been obtained by the inversion of the conductance matrix G. However, like G, it can equally well be obtained from systematic measurement of the component, but using current as excitation. Such a procedure requires a model of the component which takes a current excitation as its argument. The definition of an appropriate function, and systematic measurements to find the resistance matrix of the function *TWOV1*, is a suitable subject for exercises.

6.6 THE TRANSISTOR

Notwithstanding earlier remarks concerning the importance of a linear description of a 3-terminal component (and the attention devoted to it in Section 6.3), the **nonlinear** relation that always exists between the voltages and currents of a 3-terminal component is often a significant feature, either because it is usefully exploited (e.g., in logic gates) or because it sets a limit to attainable circuit behaviour (e.g., distortion in a high-fidelity amplifier).

The **transistor** is the most commonly used 3-terminal component. It is fabricated from either silicon or germanium, and for this reason was once called the **semiconductor triode**. The three terminals of a transistor are called the **emitter**, **base** and **collector** as indicated in Fig. 6.10. Also shown in this figure are the voltages and currents with which we shall be concerned in our measurement and modelling of a transistor.

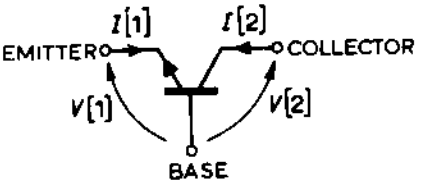


Fig. 6.10

Unlike the natural scientist, who often encounters a new phenomenon by chance, it is rare for the engineer to use a new component without previous knowledge of its existence, its physical make-up or its behaviour. For the student meeting his first transistor there are manufacturer's data sheets, curves in textbooks, and a range of theorems expounding its physical mechanism. In this way (though possibly after one or two disastrous measurements!) he will be able to propose a reasonable selection of voltage excitations to be used in exploring the nature of the transistor. To illustrate his possible approach, we use as a model of the transistor the function *TRANS* which takes, as its single argument, an array of voltage excitation whose first dimension is 2 (since the transistor has 3 terminals). The vectors `V1←¯.1 ¯.13 ¯.16 ¯.19` and `V2←0 1 5 20` respectively define the values of *V*[1] and *V*[2] to be used in the systematic measurement of the transistor. The familiar 3-dimensional array of voltage excitation:

```
      □←AV←V1 PP V2
¯0.1     ¯0.1     ¯0.1     ¯0.1
¯0.13    ¯0.13    ¯0.13    ¯0.13
¯0.16    ¯0.16    ¯0.16    ¯0.16
¯0.19    ¯0.19    ¯0.19    ¯0.19

 0        1        5        20
 0        1        5        20
 0        1        5        20
 0        1        5        20
```

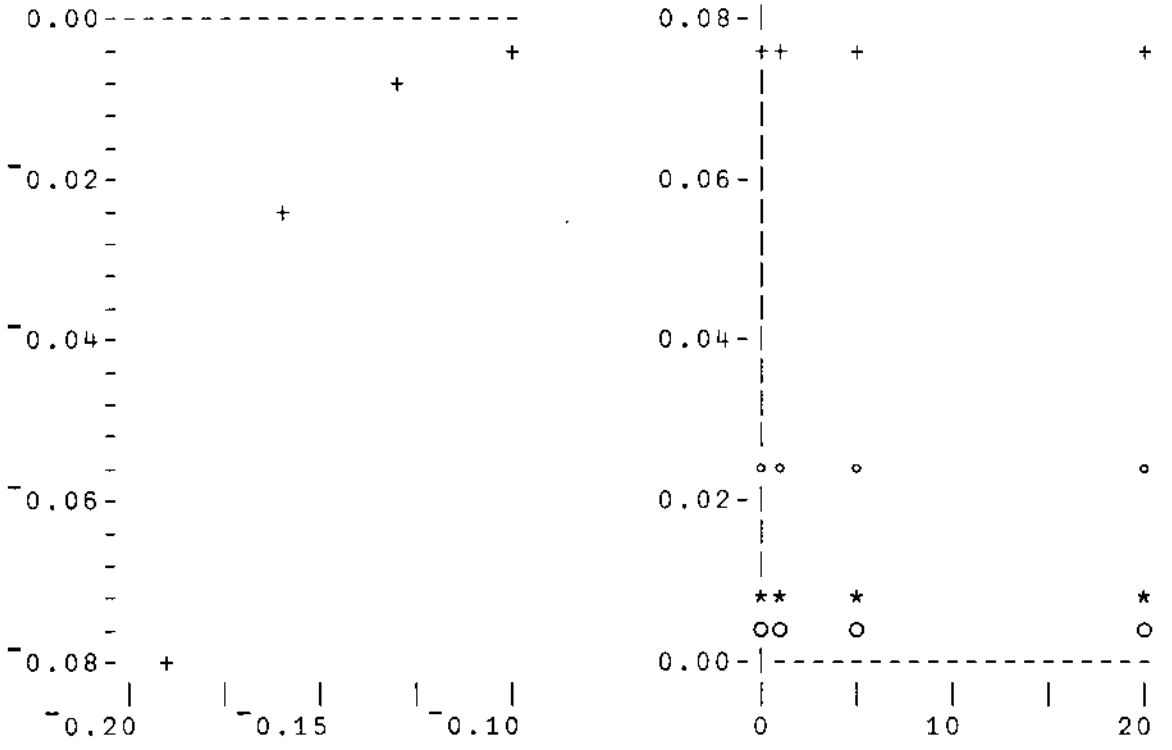
when employed as the argument of *TRANS* returns the corresponding 3-dimensional array *AI* of current response:

```
      □←AI←TRANS AV
¯0.00214    ¯0.00218    ¯0.00218    ¯0.00218
¯0.00721    ¯0.00725    ¯0.00725    ¯0.00725
¯0.024      ¯0.0241     ¯0.0241     ¯0.0241
¯0.0799     ¯0.0799     ¯0.0799     ¯0.0799

 0.00204     0.00209     0.00209     0.00209
 0.00685     0.0069      0.0069      0.0069
 0.0228      0.0229      0.0299      0.0299
 0.0759      0.0759      0.0759      0.0759
```

Plots of *I*[1] and *I*[2] versus *V*[1] and *V*[2]:

```
      20 20 PLOT AI[1;;] VS V1   20 20 PLOT(⍉AI[2;;]) VS V2
```



certainly agree, in form, with a typical manufacturer's transistor data sheet. Clearly, the function embodied in *TRANS* is not linear and homogeneous.

A more detailed examination of the effect of *V*[1] and *V*[2] upon *I*[1] and *I*[2] reveals, in those cases where the effect is significant, a roughly exponential dependence of current upon voltage:

```
      AI[1;2;]
¯0.00721 ¯0.00725 ¯0.00725 ¯0.00725
      AI[1;;2]
¯0.00218 ¯0.00725 ¯0.0241 ¯0.0799
      AI[2;2;]
0.00685 0.0069 0.0069 0.0069
      AI[2;;2]
0.00209 0.0069 0.0229 0.0759
```

Both *I*[1] and *I*[2] appear to vary exponentially with *V*[1] but are barely affected by the voltage *V*[2], at least for the range of excitation used in the measurements.

Since we suspect an approximately exponential relation between voltage excitation and current response, and especially because the transistor is a semiconductor device, we recall the approach adopted in modelling a semiconductor diode in Chapter 2. There, the relation *I*←*IS*×¯1+*40×*V* was re-expressed as:

```
      I←IS×X←TRP V                                    (6.11)
```

a relation involving an auxiliary quantity *X* to which *I* is **linearly** related. Our initial exploration suggests that a similar relation

```
      I←M+.×TRP V                                     (6.12)
```

where *I* and *V* are 2-element vectors and *M* is a 2 by 2 matrix, might well apply to the transistor.

Before testing this idea, however, we note that the values selected for *V*[1] and described by the vector *V*1

```
      V1
¯0.1 ¯0.13 ¯0.16 ¯0.19
```

are **negative**, and of increasing magnitude as we read from left to right. Since *I*[1;;2] and *I*[2;;2] similarly increase in magnitude (see above), it would appear that the transformation upon *V* should involve the calculation *-*V*, rather than **V* as in the function *TRP*. We therefore define a new transformation *TRM*

```
    ∇ R←TRM V
[1]   R←¯1+*-40×V
    ∇
```

and, in place of (6.12), explore the validity of

```
      I←M+.×TRM V                                     (6.15)
```

as a model of the function *TRANS*.

We arbitrarily select a 2 by 2 matrix of voltage excitation (i.e., two pairs of $V[1]$ and $V[2]$) and compute both the corresponding current response and the associated matrix *TRM V*

```
      V←-2 2ρ.1×4?10          I←TRANS V                X←TRM V
      V                       I                        X
¯0.1  ¯0.3            ¯1.01E6  ¯5.50E7     5.36E1   1.63E5
¯0.6  ¯0.7            ¯1.32E6  ¯7.23E7     2.65E10  1.45E12
```

Since the relation we seek is *I←M+.×TRM V* the value of *M* is computed in exactly the same way as was the matrix *G* appearing in the relation *I←G+.×V* (equation 6.19):

```
      ⌹←M←I+.×⌹X
¯4.00E¯5   3.80E¯5
 3.80E¯5  ¯5.00E¯5
```

The existence of this one matrix does not, of course, imply success in our search for a model of the function *TRANS*. Nevertheless, the same calculation carried out with different voltage excitation matrices yields the same value for *M*, and we therefore define a tentative model, *TRANSISTOR*, of the function *TRANS*:

```
    ∇ I←TRANSISTOR V;M
[1]   M←2 2ρ1E¯6×(¯40 38 38 ¯50)
[2]   I←M+.×¯1+⍟-40×V
    ∇
```

Use of the 3-dimensional voltage excitation array *AV* generated earlier allows the equivalence of the two functions to be tested:

```
      ∧/,1E¯6>|(TRANS AV)-TRANSISTOR AV
1
```

Within the accuracy (1 microamp) demanded, *TRANSISTOR* is an acceptable model of the function *TRANS*. The function which is here given the name *TRANSISTOR* is, in fact, known as the Ebers-Moll model of a transistor, and was originally deduced from consideration of the physical mechanism of the transistor.

6.7 POWER IN A 3-TERMINAL COMPONENT

The total power supplied to a 3-terminal component can be expressed by considering the component to offer three pairs of terminals into or out of which power can flow (Fig. 6.11). With reference to this figure, the total power supplied to the component can be expressed as

```
      P←(IA×VA)+(IB×VB)+(IC×VC)
```

However, by invoking Kirchhoff's laws, the voltages and currents in the above relation can be expressed in terms of the voltages and currents we have been working with (Fig. 6.12), so that *P* can equivalently be expressed in the form:

```
P←(I[1]×V[1])+(I[2]×V[2])
```

or, more concisely, as `P←+/I×V` or as `P←I+.×V`. We recall that the power supplied to a 2-terminal component was expressed in the form `P←I×V`.

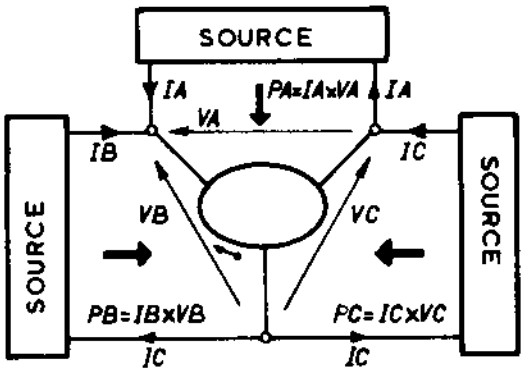


Fig. 6.11

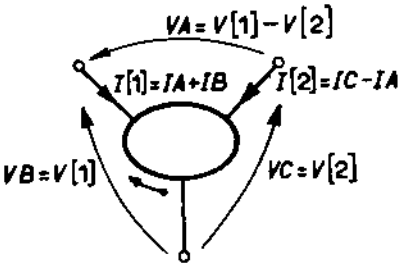


Fig. 6.12

For a single voltage excitation `V←2 7` of the component *TWOV*1, and its associated current response `I←TWOV1 V`, the power supplied to the component is simply computed

```
□←P←I+.×V
166
```

In attempting to establish the overall power behaviour of a component, however, a range of excitation may be explored. For example, if the same voltage excitation

```
V←V1 PP V2
```

used previously is applied to the transistor, yielding a current response *TRANS V*, then a simple element-by-element product of *V* and *TRANS V*, followed by sum reduction over the first dimension (to add `(I[1]×V[1])` to `(I[2]×V[2])` for each excitation) yields a matrix describing the power supplied to the component for each of the different excitation conditions:

```
      +/[1] (TRANS V) ×V
 0.000214      0.0023      0.0107      0.042
 0.000937     0.00784      0.0354      0.139
  0.00385      0.0267       0.118      0.461
   0.0152      0.0911       0.395       1.53
```

The display of this matrix is not required if it is merely the passivity or activity of the component that is of interest: in this situation the test

```
      ∧/,0≤P←+/[1]I×V
1
```

is sufficient to establish that the transistor is passive.

6.8 CIRCUITS AS MULTI-TERMINAL COMPONENTS

There is nothing special about the 3-terminal components we have just measured and modelled in this chapter, so it is important to see how our conclusions can be generalized to encompass components possessing any number of terminals. As one check, of course, our generalization should always include the case of the 2-terminal component.

For a 3-terminal component we deduced, from Kirchhoff's laws, that only two of the three terminal currents, and two of the three terminal-pair voltages, need be considered in the course of measurement and modelling. For an N-terminal component similar reasoning shows that only N-1 terminal currents need be considered; application of Kirchhoff's current law will predict the Nth. Similarly, if one terminal is chosen as the voltage reference point, it is necessary only to consider the voltage associated with the other N-1 terminals with respect to this reference since, by invoking Kirchhoff's voltage law, knowledge of these voltages is sufficient to calculate the voltage between <u>any</u> pair of terminals. Usually (as in Fig. 6.13), the terminal chosen as the point of voltage reference (and so indicated by a dot) is the one whose current is not measured.

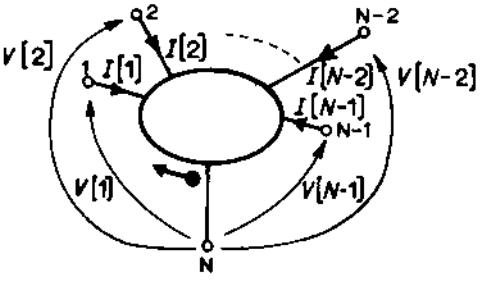


Fig. 6.13

Why is it important to consider N-terminal components? Because a <u>circuit</u> - that is, a system of interconnected components - is most profitably regarded as a single <u>component</u> having as many terminals as there are connection points. Thus, <u>what we have learned about 2- and 3-terminal components is immediately and usefully applicable, without further modification, to N-terminal circuits</u>. The importance of this idea - that circuits can be considered as components - cannot be overstated, and will be demonstrated repeatedly in later chapters.

It is important to mention, at this point, one essential difference between the consideration of an *N*-terminal component and an *N*-terminal circuit. In studying the components *TWOV1* and *TRANS* we approached the task with no knowledge of the nature of these functions, and had to establish their description by means of measurements. In contrast, if the components of a circuit are linear and homogeneous, it is known in advance that the circuit will be linear and homogeneous and, if the parameters of the constituent components are known, it is possible to calculate (rather than measure) the conductance or resistance matrix of the circuit. Apart from the absence of the burden of measurement, however, the *N*-terminal component and the *N*-terminal circuit are functionally the same. Indeed, for a linear homogeneous component, the heading '3-terminal component' appearing in many of the tables of this chapter can be replaced by "*N*-terminal component or circuit", in which case the matrices and vectors appearing in the table will have appropriate dimensions.

The next two chapters are devoted to the problem of calculating the conductance matrix of a circuit from knowledge of its constituent components and the manner in which they are interconnected. For simplicity, however, we shall consider only circuits containing linear homogeneous components and defer the treatment of nonlinear components to Chapter 11.

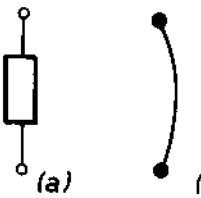
CHAPTER 7

TOPOLOGY

Three relations govern the behaviour of an electric circuit. One is the set of voltage-current relations describing the individual components, and the form that these assume has been discussed in Chapters 2 and 6. The other two sets of relations, called Kirchhoff's laws, are the subject of the present chapter. These laws were discussed in some detail in Chapter 3, but will here be placed on a more general basis.

7.1 KIRCHHOFF'S LAWS

Kirchhoff's current law concerns the currents flowing in a circuit, and is unconcerned with the voltages as well as with the relations between voltage and current imposed by the components. Similarly, Kirchhoff's voltage law is concerned solely with the relation between the various voltages in a circuit, and not with the nature of the components. Indeed, to emphasise the fact that component characteristics are immaterial when discussing Kirchhoff's laws, we introduce a new means of representing a component. A 2-terminal component (Fig. 7.1a), for example, is represented by a line whose two extremities are indicated by large dots (Fig. 7.1b). Such a representation preserves the two properties of a component that are pertinent to the present discussion:



A 2-terminal component | Its branch/node representation

Fig 7.1

1. A single identifiable current flows through the component;

2. There are exactly two points between which a voltage can be measured.

To the line, whose shape and position are immaterial, we give the name branch. The large dots which identify the points between which voltage can be measured, and which

indicate where the component can be joined to others, are called nodes. To explore the implications of this representation we consider three 2-terminal components, numbered 1, 2 and 3, connected as shown in Fig. 7.2a, and represented by Fig. 7.2b. This latter figure is sufficient to show that the three component currents all flow into the same point and, therefore, must sum to zero in accordance with Kirchhoff's current law. Similarly, the representation (Fig. 7.2c) of the 'loop' connection of three components, 1, 2, and 3, is sufficient to establish the fact that the three component voltages must sum to zero in accordance with Kirchhoff's voltage law. We see, therefore, that if only Kirchhoff's laws and the constraints they impose are of interest, then the branch/node representation of a component is an adequate one. Because it allows us to concentrate on the way that components are connected, it can yield better insight into the consequences of this interconnection.

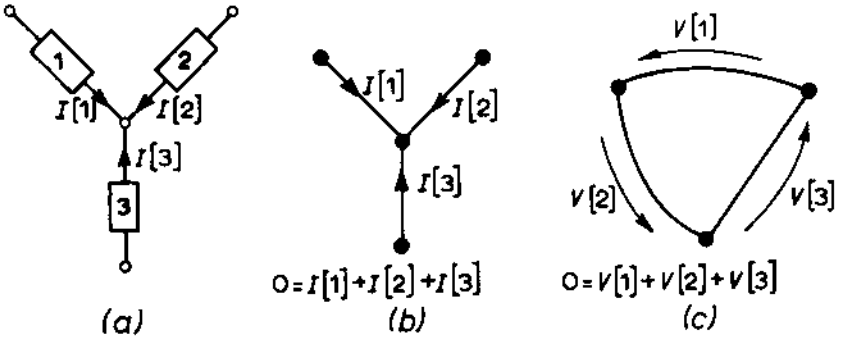


Fig. 7.2

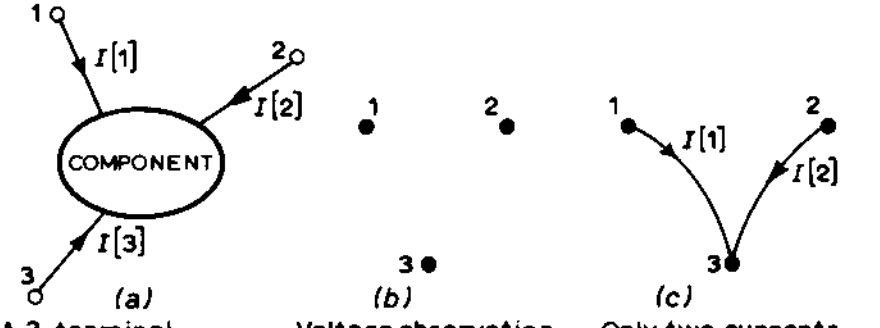


Fig. 7.3

For a 3-terminal component (Fig. 7.3a) exactly the same approach can be employed to represent the available current paths and the points between which voltage can be measured. Here there are three such points, so the representation of the component must in any event contain three nodes (Fig. 7.3b). In the previous chapter it was

decided that consideration need be given to only two of the three terminal currents, since the third can always be found by the application of Kirchhoff's current law; in our representation of the 3-terminal component (Fig. 7.3c) we therefore include only two branches along which the currents $I[1]$ and $I[2]$ flow. That such a representation is adequate is shown by the example of Fig. 7.4a; from this representation we conclude, from the application of Kirchhoff's current law, that $I[4]=I[1]+I[2]$. The fact that the component current $I[3]$ (see Fig. 7.3a) did not appear in the representation is immaterial, since none of our calculations need involve this current; recall that it was shown in the previous chapter that it is sufficient to consider just two (here $I[1]$ and $I[2]$) of the three currents associated with a 3-terminal component. Another pair of currents - say $I[1]$ and $I[3]$ - could equally well have been chosen, in which case the corresponding representation would be as shown in Fig. 7.4b. In this case the connection shown in Fig. 7.4a would be represented as shown in Fig. 7.4c, from which we would conclude, from the application of Kirchhoff's current law, that $I[3]=-I[4]$.

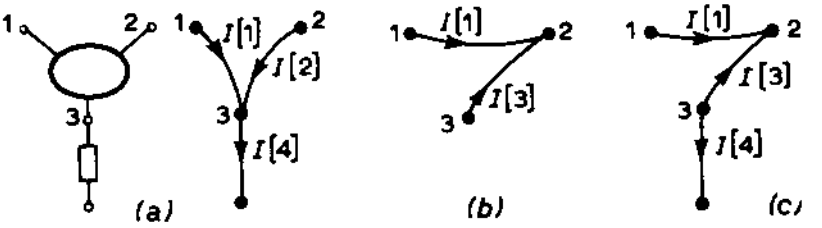


Fig. 7·4

In a similar manner we observe that knowledge of the two voltages associated with the two branches is sufficient to determine any other voltages. For components having more than three terminals a general rule is easy to discern. The number of nodes is the same as the number of terminals that the component possesses. Since there are N nodes only N-1 currents and N-1 voltages need be considered. The most straightforward representation of an N-terminal component is therefore as shown in Fig. 7.5.

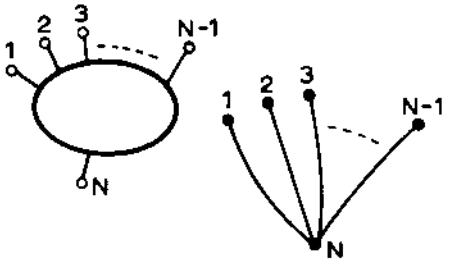
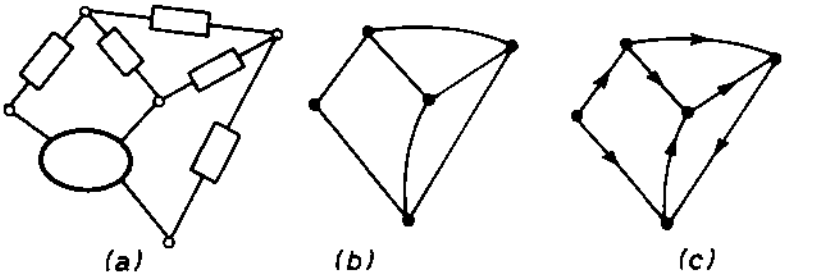


Fig. 7·5

An N-terminal component and its branch-node representation.

7.2 DIRECTED GRAPH

If we are only concerned with the relations between circuit currents and the relations between circuit voltages, the circuit can be re-drawn using the branch/node representation for each component. In this way, Fig. 7.6b represents the circuit of Fig. 7.6a. The collection of nodes and branches in Fig. 7.6b is called the graph of the circuit. Thus, the graph describes only the topology of the circuit - that is, the manner in which the components are connected together. It yields no information about the characteristics of the individual components. It may, in fact, be likened to the familiar map of the London Underground System, Fig. 7.7, which indicates the stations linked by the various routes, but would be useless for surface navigation by car since it carries no precise geographic information.



(a) A circuit — (b) Its branch-node representation — (c) Its directed graph

Fig. 7.6

As in Chapter 2, we assign a reference direction to each branch, and in this way obtain the directed graph of Fig. 7.6c. The arrows serve to define the reference direction for both current and voltage in accordance with the convention introduced in chapter 2. Finally, to allow the identification of branches and nodes, we number them. This numbering can be arbitrary, but later discussion is eased considerably if the numbering is consecutive, beginning with 1 (Fig. 7.8). We shall adopt the convention of circling branch numbers in order to distinguish them from node numbers. It should be noted that, up to this point, we have merely 'labeled' the circuit graph for convenience of discussion: there is no more information in Fig. 7.8 than there was in Fig. 7.6a.

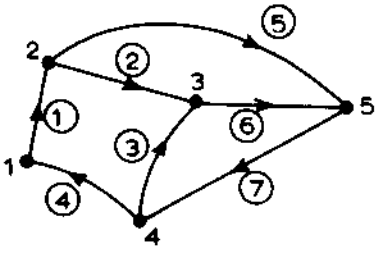
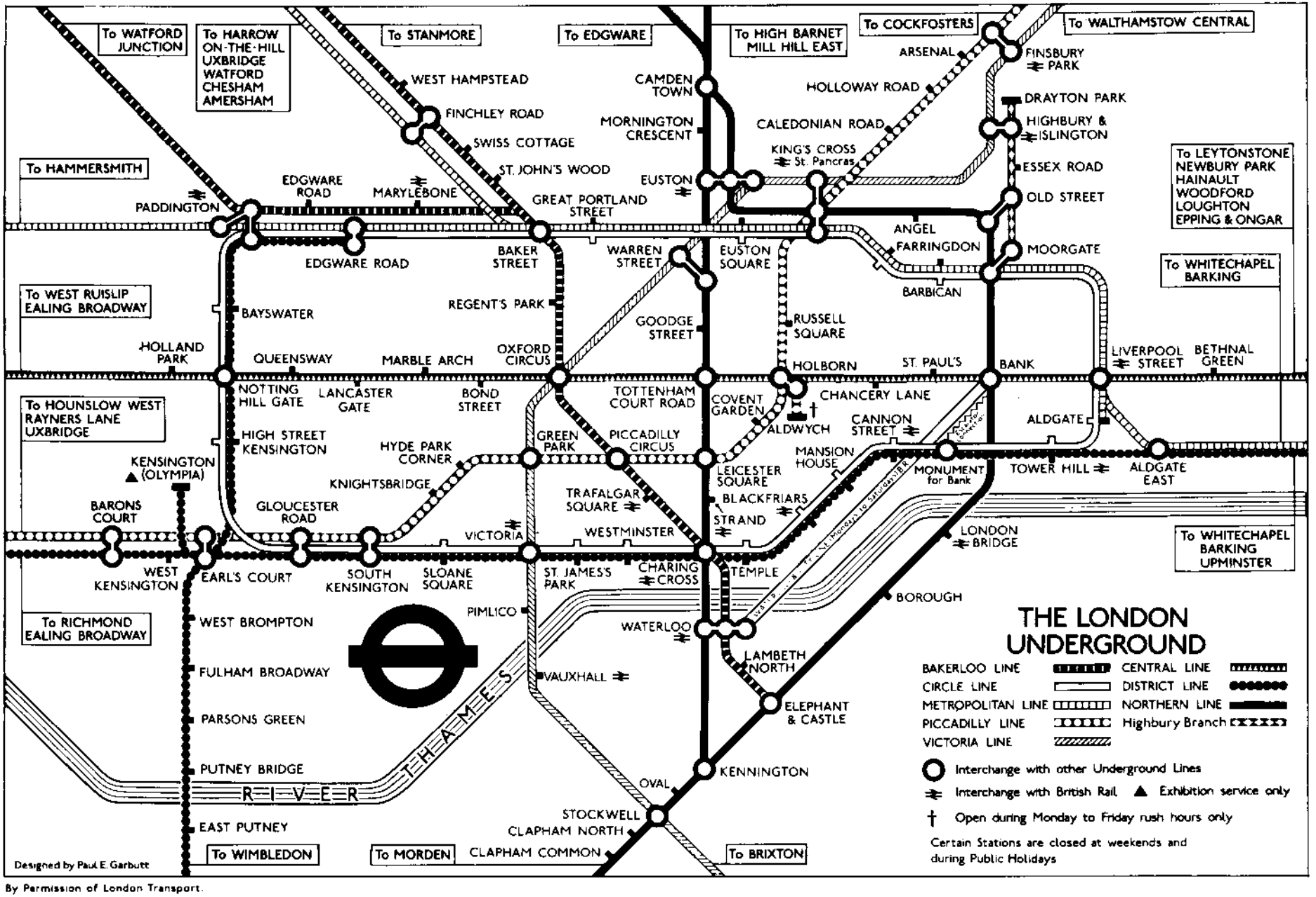


Fig. 7.8
A directed graph of the circuit of Fig. 7.6a



By Permission of London Transport.

Fig. 7.7

How can a directed graph be described? The method adopted here - and many others are possible - is illustrated by the branch connection matrix *M*

```
      M
1 2 4 4 2 3 5
2 3 3 1 5 5 4
```

which describes the directed graph of Fig. 7.8. The first and second rows of *M* are, respectively, a list, in the order of the branch numbering, of the nodes out of and into which the arbitrarily assigned reference currents flow. For example, the third column of *M* tells us that branch 3 is connected between nodes 4 and 3, with the current reference arrow pointing away from node 4 and towards node 3.

Since the matrix *M* is a complete description of the directed graph of Fig. 7.8 we can expect to be able to deduce, from *M*, the numbers of nodes and branches. The number of nodes is the maximum over the ravel of *M*, while the number of branches is equal to the second dimension of *M*:

```
      ,M
1 2 4 4 2 3 5 2 3 3 1 5 5 4
      □←N←⌈/,M
5
      ρM
2 7
      □←B←¯1↑ρM
7
```

The method of computing *N* from *M* is only valid, of course, if the nodes are numbered consecutively, beginning at 1.

7.3 THE INCIDENCE MATRIX

Although the branch connection matrix *M* is easy to write down from inspection of the directed graph, an alternative (equivalent) description is used almost exclusively in later calculations of circuit behaviour. Why is this so?

The answer is obtained by reference to our earlier discussion of Kirchhoff's current law in Section 3.1 of Chapter 3. To express the consequence of this law at a particular junction (node) - for example node 3 of Fig. 7.8 - we generate the element-by-element product of two vectors. The first of these:

```
      IB←13 ¯8 ¯11 13 21 ¯19 2
```

is a vector of branch currents which obeys Kirchhoff's current law* for the graph of Fig. 7.8. The other,

```
      A3←0 ¯1 ¯1 0 0 1 0
```

(which is particular to the node in question) contains the entries 1, ¯1 and 0 according to whether each branch reference current flows out of the node, into the node, or is unassociated with the node; it is called the incidence vector. The element-by-element product of these two vectors yields the branch currents which flow away from the node:

```
      A3×IB
0 8 11 0 0 ¯19 0
```

and the sum reduction over this vector

```
      +/A3×IB
0                                                    (7.1)
```

yields a result which is consistent with Kirchhoff's current law at node 3. Equation (7.1) is a statement of Kirchhoff's current law for node 3 of Fig. 7.8, and therefore a series of such statements - one for each node in the circuit - can summarize the constraints imposed by Kirchhoff's current law upon the whole circuit.

An alternative and more compact summary is a single equation involving a matrix *A* which, for the directed graph of Fig. 7.8, is

```
      A
 1   0   0  ¯1   0   0   0
¯1   1   0   0   1   0   0
 0  ¯1  ¯1   0   0   1   0
 0   0   1   1   0   0  ¯1
 0   0   0   0  ¯1  ¯1   1
```

The rows of *A* are, in order, the incidence vectors previously associated with the nodes; for example, row 3

```
      A[3;]
0 ¯1 ¯1 0 0 1 0
```

is identical with the vector *A3* introduced into the previous discussion. In other words, *A* is an *N* by *B* matrix whose elements are 1, ¯1 or 0, respectively, according to whether the reference current of a branch leaves, enters or is not

---

* These are not necessarily the actual currents flowing in the circuit.

associated with a node. It follows that the matrix product of *A* with the vector *IB* of branch currents

```
      A+.×IB
0 0 0 0 0
```

yields a vector of zeros which is consistent with the application of Kirchhoff's current law to each node of the circuit. It follows that the relation

```
      ∧/0=A+.×IB
```

is a compact statement of Kirchhoff's current law. The matrix *A* is called the incidence matrix of the circuit (or, more accurately, of the circuit's directed graph) or, more precisely, the node-branch incidence matrix. Like the matrix *M* it is a complete description of a directed graph.

The relation between the branch connection matrix and the incidence matrix can be explored as follows by means of the previous example. The test

```
      2=M[1;]
0 1 0 0 1 0 0
```

to see which branches have node 2 as their 'out of' node can be extended to all nodes, of which there are *N*,

```
      (⍳N)∘.=M[1;]
1 0 0 0 0 0 0
0 1 0 0 1 0 0
0 0 0 0 0 1 0
0 0 1 1 0 0 0
0 0 0 0 0 0 1
```

to generate the 1 elements of the incidence matrix. A similar calculation involving the second row of *M* (the 'into' nodes) yields a matrix whose 1 elements occupy the same locations as the ¯1 elements of the incidence matrix:

```
      (⍳N)∘.=M[2;]
0 0 0 1 0 0 0
1 0 0 0 0 0 0
0 1 1 0 0 0 0
0 0 0 0 0 0 1
0 0 0 0 1 1 0
```

Clearly, the difference of these matrices yields the required incidence matrix. Since we shall frequently have need to generate the incidence matrix of a circuit we refer to the above relation between *M* and *A* in defining a function

which takes the branch connection matrix as its argument and produces the N by B incidence matrix:

```
     ∇ A←INCID M;N
[1]    A←((⍳N)∘.=M[1;])-(⍳N←⌈/,M)∘.=M[2;]
     ∇
```

7.4 KIRCHHOFF'S CURRENT LAW

When a circuit is put to practical use, one or more sources of excitation will be connected to it. In this case it is usually helpful to make a distinction between the currents flowing in the components, and the excitation currents injected into the nodes from external sources. Thus, if the excitation shown in Fig. 7.9 is applied to the circuit of Fig. 7.6 (whose directed graph is Fig. 7.8) there are two current vectors of interest. One,

```
      IB←5 ¯3 4 ¯1 8 1 7
```

is the vector of branch currents. The other, having as many elements as there are nodes, is the vector I of <u>nodal excitation currents</u>:

```
      I
6  0  0  ¯4  ¯2
```

The same calculation carried out above, and which computes the total current flowing away from each node <u>along the branches</u>, does not now yield a vector of zeros.

```
      A+.×IB
6  0  0  ¯4  2
```

As required by Kirchhoff's law, these currents are equal to the nodal excitation currents. The application of Kirchhoff's current law to all nodes is therefore summarized concisely by the relation

```
      I←A+.×IB                                    (7.2)
```

Note that the elements of I sum to zero

```
      +/I
0
```

as would be expected by applying Kirchhoff's current law to the generalized node (recall Chapter 3) defined by the dashed line in Fig. 7.9, whereas the same property need not apply to the vector IB:

```
      +/IB
21
```

The above calculations, using the example of Fig. 7.5, have illustrated the importance of the incidence matrix in the concise statement of Kirchhoff's current law for an entire circuit.

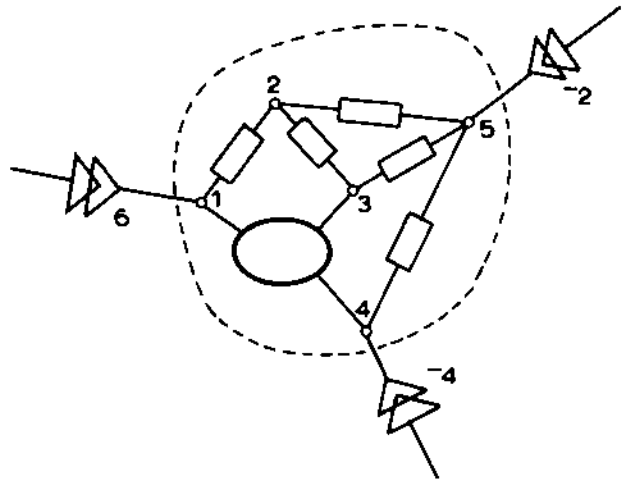


Fig. 7.9 Current excitation of the circuit of Fig. 7.6

7.5 KIRCHHOFF'S VOLTAGE LAW

Can Kirchhoff's voltage law also be expressed concisely in terms of the incidence matrix? To answer this question, we choose as an example the circuit whose directed graph is shown in Fig. 7.10. This 6-branch graph is described as follows:

```
      M←⍉6 2⍴1 2 4 1 3 1 4 3 3 2 2 4
      ⎕←A←INCID M
 1 ¯1 ¯1  0  0  0
¯1  0  0  0 ¯1  1
 0  0  1 ¯1  1  0
 0  1  0  1  0 ¯1
```

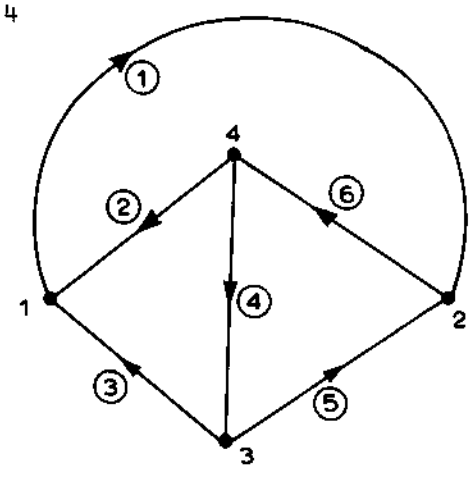


Fig. 7.10 The directed graph of a circuit.

Each branch voltage is the difference between the potentials of the nodes to which the branch is connected. For each branch, the appropriate column of A identifies not only the nodes to which it is connected, but also the reference direction. For example, since $A[;4]$ is $0\ 0\ {}^{-}1\ 1$, it follows that branch 4 joins nodes 3 and 4 and that the potential of node 3 must be subtracted from that of node 4 to yield the voltage across branch 4. Therefore, if

the nodal potentials (commonly called the **nodal voltages**) are described by the vector *V*

```
      V←1 4 ¯2 7
```

the sum reduction of the element-by-element product of *V* and *A*[;4] yields a value for the branch voltage *VB*[4]:

```
      +/V×A[;4]
9
```

A similar method of computing a component voltage from the vector of terminal potentials was discussed in Section 3.3 of Chapter 3. The extension of this calculation to yield the vector *VB* of branch voltages is again given by an matrix product

```
      □←VB←V+.×A
¯3  6  ¯3  9  ¯6  ¯3
```

Though it follows that a concise expression of Kirchhoff's voltage law for an entire circuit is

```
      VB←V+.×A                                    (7.3)
```

we shall usually employ the equivalent expression

```
      VB←(⍉A)+.×V                                 (7.4)
```

involving the transpose of the incidence matrix. This form places the argument on the right, in the same position occupied by the argument *IB* in equation (7.2).

To show that the reference point for nodal voltages is irrelevant to the calculation of branch voltages - since the latter are **differences** between nodal voltages - we may change the voltage of the reference point for nodal voltages by 100 volts and note that the branch voltage vector is unchanged:

```
      (⍉A)+.×V+100
¯3  6  ¯3  9  ¯6  ¯3
```

Although an arbitrary choice of reference points can be made, we often choose a circuit node as the reference point for nodal voltages. If node 2 is so chosen, we must modify the nodal voltages so that *V*[2] is zero:

```
      V←V-V[2]
      (⍉A)+.×V
¯3  6  ¯3  9  ¯6  ¯3
```

These calculations confirm that such a choice is valid. Nevertheless, the fact that one of the nodal voltages is now zero by definition must not lead us to forget to include it in our vector of nodal voltages.

7.9 KIRCHHOFF'S LAWS FOR CHANGES IN VOLTAGE AND CURRENT

The discussion of Chapter 5 has shown that under certain circumstances, and especially when signals are present, it may be convenient to give separate consideration to the constant and time-varying components of a voltage or current source. In particular, in computing the signal behaviour of a circuit, we are interested only in the changes in the voltages and currents associated with the circuit. The natural question to ask is, "do these changes obey Kirchhoff's laws?" In other words, do the following relations hold (cf. equations (7.2), (7.3))?

```
ΔI←A+.×ΔIB
ΔVB←ΔV+.×A
```

The answer to this question forms the subject of an exercise.

CHAPTER 8

CIRCUIT DESCRIPTION

Let us, for a moment, stand back from the detail of the past few chapters and briefly review the original motivation for our study of circuit theory.

It was explained in Chapter 1 that the specification for a circuit is laid down by the system designer who, in effect, describes the desired functional relation between an excitation and a response. The excitation may be described by a vector I of injected nodal currents* and the response by a vector V of nodal response voltages (Fig. 8.1)**. We see that the system designer is essentially describing the circuit as a component, a concept discussed in Section 6.8 and illustrated in Fig. 6.13.

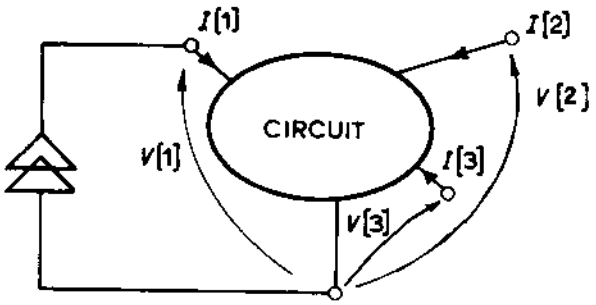


Fig. 8·1

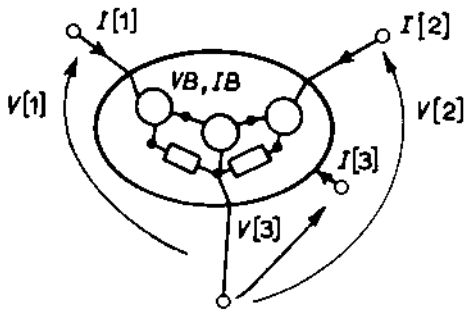


Fig. 8·2

---

* It may be argued that circuits are never excited by ideal current sources. Nevertheless, if actual sources are represented by their Norton models, and the internal resistances considered to be part of the circuit, this circuit is then excited by ideal current sources.

** Often, as in Fig. 8.1, only one or two elements of I may be non-zero. Similarly, the system designer may be interested in only one voltage, and hence will specify only one element of V.

The circuit designer is concerned with choosing components and connecting them together (Fig. 8.2) in such a way that the stipulated excitation *I* leads to the required response *V*. In the course of the design, therefore, and also in the circuit analysis employed to check the design, the circuit designer will have to be concerned with the currents and voltages associated with each component as well as with the excitation currents and response voltages. More precisely, he must be concerned with the vector *VB* of branch voltages, the vector *IB* of branch currents, the vector *I* of nodal currents and the vector *V* of nodal voltages. Since the system designer is concerned only with the vectors *I* and *V* it follows that the circuit designer, after proposing what he feels is a satisfactory circuit, must discover the relation that the circuit imposes between *I* and *V*. In other words, from knowledge of the components and their interconnection, he must be able to describe the circuit as a component. Let us see how this can be done.

8.1 THE BRANCH CONDUCTANCE MATRIX

The behaviour of electric circuits is governed by three sets of relations, of which we have already expressed two in array notation,

Kirchhoff's Current Law $I \leftarrow A +.\times IB$ (8.1)

Kirchhoff's Voltage Law $VB \leftarrow V +.\times A$ (8.2)

where *A* is the circuit's incidence matrix. Appearing with the vectors *I* and *V* that are of interest to the system designer are the vectors *IB* and *VB* which must be eliminated from the final description of the circuit. To this end we now examine the relation between *IB* and *VB* for one or two circuits to see what general form the relation takes. Before starting, it is important to remind the reader that only circuits containing linear homogeneous components are considered here: nonlinear circuits are considered in Chapter 11.

If we are concerned only with the relation between *IB* and *VB* - called the branch relation - then it doesn't matter if we consider the components connected together or simply as a collection of unconnected components. We have, after all, taken account of the way they are to be connected in the incidence matrix in equations (8.1) and (8.2).

In the first example we consider a circuit composed of the four 2-terminal components which are shown in Fig. 8.3. Each component is represented by one branch, and the branches are numbered. Since there are four 2-terminal

components, and hence four branch currents and voltages, both *IB* and *VB* are 4-element vectors. They are related by the expression

```
IB←YB+.×VB
```

where *YB* is the 4 by 4 **Branch Conductance Matrix**:

```
      YB
 2  0  0  0
 0  1  0  0
 0  0  8  0
 0  0  0 10
```

The matrix *YB* is easily interpreted. The first row, associated with the current *IB*[1] in branch 1, states that *IB*[1] is found by multiplying *VB*[1] by 2, *VB*[2] by zero, *VB*[3] by zero and *VB*[4] by zero, and summing these products. The same rule applies to each row of *YB*. Row 1 is, in this way, consistent with the relation

```
IB[1]=2×VB[1]
```

imposed upon *IB*[1] and *VB*[1] by the component numbered 1. Note that the dimension of the branch conductance matrix is *B*,*B* where *B* is the number of branches.

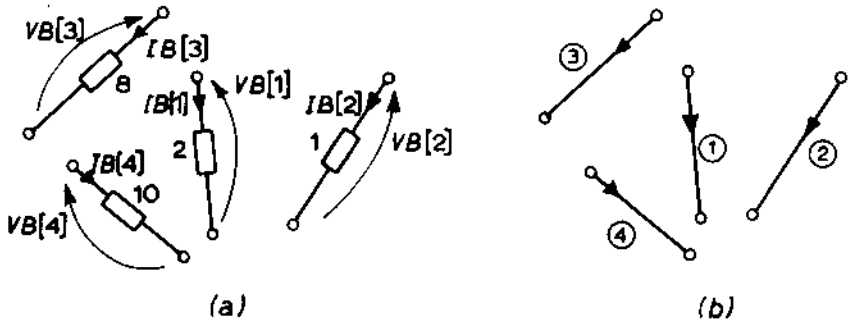


Fig. 8·3 Four 2-terminal components and their branch/node representations

To avoid tedium in the generation of the branch conductance matrix of a collection of 2-terminal components we shall define a function which takes, as its single argument, a vector whose elements are the separate branch conductances. For example, the vector *G*

```
G←2 1 8 10
```

defines the conductances of the components of Fig. 8.3 in the order of their numbering. One possible algorithm for

generating the branch conductance matrix proceeds as follows. First, the number of branches is computed

```
      □←B←ρG
4
```

Next, a B by B matrix T is formed from the vector G

```
      □←T←(B,B)ρG
2 1 8 10
2 1 8 10
2 1 8 10
2 1 8 10
```

in such a manner that subsequent element-by-element multiplication by a unit matrix of dimension B,B

```
      T×UNIT B
2 0 0  0
0 1 0  0
0 0 8  0
0 0 0 10
```

```
      ∇ U←UNIT N
[1]     U←(ιN)∘.=ιN
      ∇
```

generates the required result. Thus, a suitable function for generating the branch conductance matrix of a collection of 2-terminal components is

```
    ∇ YB←COND G;B
[1]   YB←((B,B)ρG)×UNIT B←ρG
    ∇
```

The above discussion has shown by example that each 2-terminal component gives rise to one element of the branch conductance matrix, and that this element occurs on the main diagonal. All off-diagonal elements are zero.

Nevertheless, many of the circuits that must be analyzed contain 3-terminal as well as 2-terminal components. Fig. 8.4a shows the components (in this case, each has 3 terminals) that go to make up a certain circuit, together with their conductance matrices, while Fig. 8.4b shows the choice of branches which is consistent with the reference terminals indicated in Fig. 8.4a. Individually, therefore, the components are described by

$$IX \leftarrow GX +.\times VX \qquad (8.3a) \qquad\qquad IY \leftarrow GY +.\times VY \qquad (8.3b)$$

where

```
      GX
2  ¯5
7   4
```

```
      GY
 4  10
¯1   9
```

For the collection of components in Fig. 8.4 there are four branches altogether, so that the branch current vector *IB* and the branch voltage vector *VB* must each have four elements. If *IB* is defined in terms of *IX* and *IY* by

```
IB←IX,IY
```

and *VB* similarly in terms of *VX* and *VY*

```
VB←VX,VY
```

then the relation between *IB* and *VB* is

```
IB←YB+.×VB
```

where

```
      YB
 2   ¯5   0   0
 7    4   0   0
 0    0   4  10
 0    0  ¯1   9
```

To aid interpretation of this matrix, the locations of the elements of *IB* and *VB* are shown in Fig. 8.4c for the same two components, and the new branch numbers are shown. Row 3 of *YB* shows, for example, that *IB*[3] (previously called *IY*[1]) is obtained by multiplying *VB*[3] (previously *VY*[1]) by 4 and *VB*[4] (previously *VY*[2]) by 10, and then adding these products; this is, in fact, in accordance with equation (8.3b).

For the example of Fig. 8.4 the generation of *YB* from *GX* and *GY* is a straightforward procedure. The dimension *I* of *YB* is simply related to the dimensions of *GX* and *GY* by

```
I← (⍴GX)+ ⍴GY
```

so that the sum of the appropriate overtakes of *GX* and *GY* yields the branch conductance matrix:

```
      I↑GX                    (-I)↑GY
 2   ¯5   0   0          0    0    0    0
 7    4   0   0          0    0    0    0
 0    0   0   0          0    0    4   10
 0    0   0   0          0    0   ¯1    9

      ⎕←YB← (I↑GX)+ (-I)↑GY
 2   ¯5   0   0
 7    4   0   0
 0    0   4  10
 0    0  ¯1   9
```

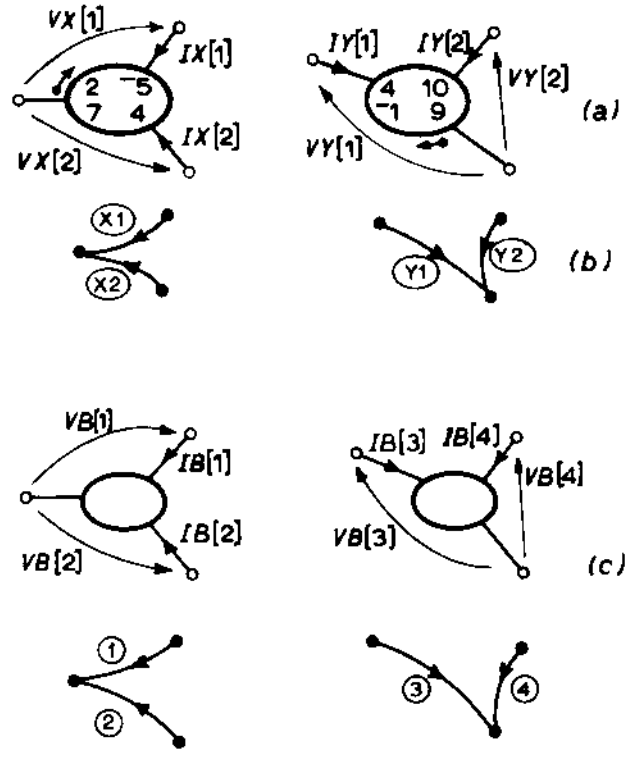

VX[1]
IX[1]
2 −5
7 4
IX[2]
VX[2]
X1
X2
IY[1]
IY[2]
VY[2]
4 10
−1 9
VY[1]
(a)
Y1
Y2
(b)
VB[1]
IB[1]
IB[2]
VB[2]
IB[3]
IB[4]
VB[4]
VB[3]
(c)
1
2
3
4

Fig. 8·4

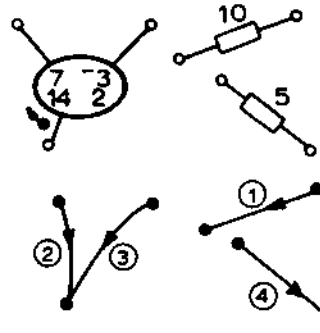

7 −3
14 2
10
5
2
3
1
4

Fig. 8·5

This procedure leads to the definition of a function *AA* by means of which the branch conductance matrix can be generated from the conductance matrices of the components:

```
     ∇ YB←A AA B;I
[1]    I←(ρA)+ρB
[2]    YB←(I↑A)+(-I)↑B
     ∇
```

Repeated application of the function *AA* would be required to compute the branch conductance matrix of a collection of more than two 3-terminal components. The procedure just described is not, of course, limited to 3-terminal components: those with more than three terminals are handled in exactly the same way.

The third and final example of a branch conductance matrix is that which describes the collection of components shown in Fig. 8.5. For the branch numbering shown*, the branch conductance matrix is

```
 10   0   0   0
  0   7  ¯3   0
  0  14   2   0
  0   0   0   5
```

as the reader can verify. The only point to note concerns the use of the function *AA* to generate the above matrix. Because it is assumed, on line [1], that *A* and *B* are matrices, the scalar conductance value of a 2-terminal component must first be converted to a 1 by 1 matrix. To cater for the situation when either or both arguments of *AA* are scalars, the function is appropriately modified:

```
     ∇ YB←X AA Y;XX;YY;I
[1]    XX←(⌈2↑1 1,ρX)ρX
[2]    YY←(⌈2↑1 1,ρY)ρY
[3]    I←(ρXX)+ρYY
[4]    YB←(I↑XX)+(-I)↑YY
     ∇
```

We test the new function with the examples of Fig. 8.3 and Fig. 8.5

```
      □←YB←2 AA 1 AA 8 AA 10
 2  0  0  0
 0  1  0  0
 0  0  8  0
 0  0  0 10
```

```
      G←2 2ρ 7 ¯3 14 2
      □←YB←10 AA G AA 5
 10   0   0   0
  0   7  ¯3   0
  0  14   2   0
  0   0   0   5
```

---

* The order in which branch numbering has been carried out in the examples has always assigned consecutive numbers to the two branches of a 3-terminal component. This is consistent, and helpful to an understanding of the branch conductance matrix, but is not necessary.

8.2 THE NODAL EQUATIONS

We have expressed, in matrix form, the three sets of relations governing the behaviour of circuits composed of linear homogeneous components*:

Kirchhoff's current Law	`I←A+.×IB`	(8.4)
Kirchhoff's voltage Law	`VB←(⍉A)+.×V`	(8.5)
The Branch Relations	`IB←YB+.×VB`	(8.6)

Elimination of *IB* and *VB* is now possible. Substitution of (8.6) in (8.4) yields

```
I←A+.×(YB+.×VB)                                    (8.7)
```

and substitution of (8.5) in (8.7) then yields

```
I←A+.×(YB+.×((⍉A)+.×V))                            (8.8)
```

a relation whose re-expression as

```
I←(A+.×YB+.×⍉A)+.×V                                (8.9)
```

is permitted by the associativity of matrix multiplication.

Expression (8.9) is the one we have been seeking because it relates the vector *I* of nodal currents to the vector *V* of nodal voltages. The relation

```
I←Y+.×V                                            (8.10)
```

is referred to as the nodal equation, and the matrix *Y* (cf. equation 8.9) is called the nodal conductance matrix, and is defined as follows:

```
Y←A+.×YB+.×⍉A                                      (8.11)
```

A number of important observations can be made:

(1) The circuit description of equation (8.10) is of the same general form as the component descriptions we have already encountered (e.g., in chapter 6), and supports the view that a circuit can be regarded as a component.

(2) The dimension of the nodal conductance matrix *Y* is *N* by *N*, where *N* is the number of nodes.

---

* But recall that (8.4) and (8.5) are valid independently of the nature of the components, i.e., (8.4) and (8.5) are also valid for nonlinear circuits.

(3) Just as the circuit is described by the components and the way they are connected together, so is *Y* a function solely of *YB* and *A*.

8.3 THE NODAL CONDUCTANCE MATRIX

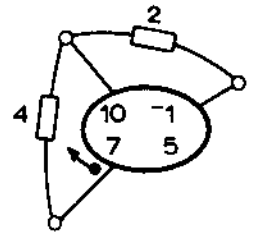
In view of the last observation it is useful to define a function which generates the nodal conductance matrix of a circuit and takes, as its arguments, the incidence matrix *A* and the branch conductance matrix *YB*:

```
    ∇ Y←A NODMAT YB
[1]   Y←A+.×YB+.×⍉A
    ∇
```

This function, together with *COND*, *AA* and *INCID* (all summarized in Table 8.1), can now be used as a building block in the generation of nodal conductance matrices. As an illustration, consider the circuit of Fig. 8.6. Its directed graph is described as follows:

```
      ⎕←M←⍉4 2⍴3 2 3 2 1 2 1 3
3 3 1 1
2 2 2 3

      ⎕←A←INCID M
 0  0 ¯1  1
¯1 ¯1 ¯1  0
 1  1  0 ¯1
```



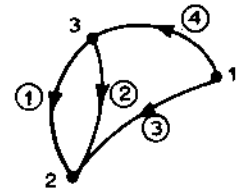


Fig. 8.6 A circuit and its directed graph

The components are described by the branch conductance matrix *YB*:

```
      □←YB←4 AA(2 2 ρ 10 ¯1 7 5)AA 2
 4   0   0   0
 0  10  ¯1   0
 0   7   5   0
 0   0   0   2
```

Use of the function *NODMAT* produces the nodal conductance matrix of the circuit:

```
      ¯□←Y←A NODMAT YB
 7 ¯12   5
¯4  25 ¯21
¯3 ¯13  16
```

In particular, the dimension ρ*Y* is 3,3, consistent with the number of circuit nodes ⌈/,*M* which is 3.

```
     ∇ Y←A NODMAT YB
[1]    Y←A+.×YB+.×⍉A
     ∇
     ∇ YB←COND G;B
[1]    YB←((B,B)ρG)×UNIT B←ρG
     ∇
     ∇ YB←X AA Y;XX;YY;I
[1]    XX←(¯2↑1 1,ρX)ρX
[2]    YY←(¯2↑1 1,ρY)ρY
[3]    I←(ρXX)+ρYY
[4]    YB←(I↑XX)+(-I)↑YY
     ∇
     ∇ A←INCID M;N
[1]    A←((⍳N)∘.=M[1;])-(⍳N←⌈/,M)∘.=M[2;]
     ∇
     ∇U←UNIT N
[1]    U←(⍳N)∘.=⍳N
     ∇
```

Table 8.1
Functions involved in the generation of nodal conductance matrices

There will be circumstances in which neither the incidence matrix nor the branch conductance matrix is of immediate interest. In this event the nodal conductance matrix can be generated directly by the procedure

```
Y←(INCID M) NODMAT 4 AA (2 2 ρ 10 ¯1 7 5) AA 2
```

As a brief illustration of the significance of the nodal conductance matrix we calculate, for the circuit of Fig. 8.6, the current response to the terminal voltage excitation shown in Fig. 8.7a. The terminal voltage excitation is described by the vector

```
      V←2 14 ¯7
```

and therefore the current response is, according to (8.10),

```
      ⎕←I←Y+.×V
¯189 489 ¯300
```

In other words, the currents flowing into terminals 1, 2 and 3 are ¯189 amps, 489 amps and ¯300 amps respectively. Conversely, if these currents were injected into the terminals from ideal current sources (Fig. 8.7b), the resulting terminal potentials would differ from each other by the same amount as in the vector V.

8.4. THE REDUCED NODAL CONDUCTANCE MATRIX

In the discussion following equation (8.11) it was remarked that the dimension of a nodal conductance matrix is N by N, where N is the number of circuit nodes. It was also pointed out that the nodal relation $I←Y+.×V$ can be regarded as the description of a component, in which case Y would be its conductance matrix. But in Chapter 6 it was established that an N-terminal component is described, not by an $N×N$ matrix, but by an $(N-1)$ by $(N-1)$ conductance matrix. Can we resolve the apparent discrepancy?

Let us examine the nodal conductance matrix of the circuit of Fig. 8.6:

```
      Y
 7 ¯12   5
¯4  25 ¯21
¯3 ¯13  16
```

Examination of this matrix soon reveals an interesting property; each row and each column sums to zero:

```
      +/[1]Y
0 0 0
      +/[2]Y
0 0 0
```

Further examples would show that this property is shared by all nodal conductance matrices. The property is easily explained by considering two different excitation conditions for the circuit. The first condition is shown in Fig. 8.8; a source of current excitation, as yet unknown in value, is applied to each terminal. We wish to know the values that

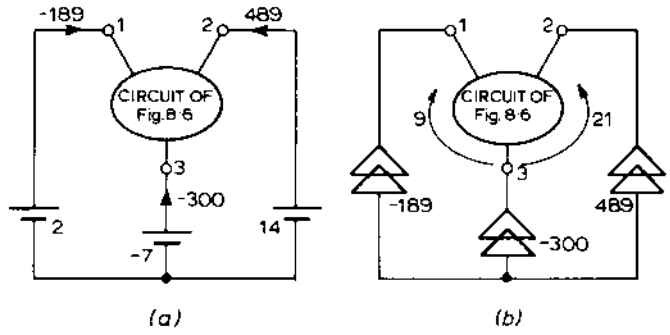


Fig. 8·7

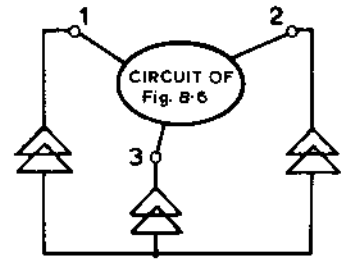


Fig. 8·8

these currents must have if all the resulting terminal potentials are to be equal. The answer is easily computed by assuming an arbitrary value for the equal terminal potentials

```
      V←3 ⍴163.5
```

and computing the corresponding vector of nodal currents:

```
      ⎕←I←Y+.×V
0 0 0
```

Different choices of V (with all elements equal) yield the same result: that the terminal potentials are equal only if no currents are injected into the terminals. This result is reasonable on physical grounds: if no currents are injected, then no voltages will be developed between the terminals of the circuit, since the circuit does not contain any independent sources: it is a 'dead' circuit in this respect. Since each of the elements of I must be zero when the elements' of V are equal, it follows that each row of Y must sum to zero.

The second excitation condition is shown in Fig. 8.9, where all the terminal voltages except one are set to zero. Here we have chosen the voltage of terminal 2 to be the only non-zero voltage, and have assigned it an arbitrary value:

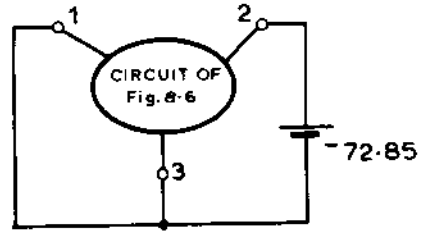


Fig. 8·9

```
      V←0 ¯72.85 0
```

The currents flowing into each of the terminals are described by the vector I:

```
      ⎕←I←Y+.×V
874 ¯1.82E3 947
```

We are not surprised to find that the currents have non-zero values, or that they obey Kirchhoff's current law:

```
      +/I
0
```

Since $V[1]$ and $V[2]$ are zero the same sum (i.e., $I[1]+I[2]+I[3]$) can alternatively be found by computing

```
      (Y[1;2]+Y[2;2]+Y[3;2])×V[2]
0
```

or

```
      +/Y[;2]×V[2]
0
```

If the result must be zero <u>whatever the vaue of</u> $V[2]$, it follows that column 2 of Y must sum to zero. By considering the remaining terminals in the same way it follows that every column of Y must sum to zero.

Do the zero row-sum and zero column-sum properties imply some redundancy in the nodal conductance matrix? In one sense, yes; what is happening is that the procedure

```
      I←Y+.×V
```

computes <u>all</u> the terminal currents of a circuit, whereas we know that we can deduce <u>one</u> of these by invoking Kirchhoff's current law (e.g., `I[3]←-+/I[1],I[2]` in Fig. 8.7). Let us, therefore, refer to the circuit of Fig. 8.7 and delete the row of Y associated with this current (say, $I[3]$)

```
      □←X←¯1 0↓Y
  7 ¯12   5
 ¯4  25 ¯21
```

Matrix multiplication by the vector of terminal voltages `V←2 14 ¯7` (the excitation of Fig. 8.7) still yields the correct values for $I[1]$ and $I[2]$:

```
      X+.×V
¯189 489
```

But we can make a further reduction in the size of the matrix needed to describe the circuit. If we choose one of the circuit terminals as the reference point for terminal voltages, the corresponding element of V is zero. Since this element will multiply the appropriate column of Y, we might as well eliminate the column, and reduce the dimension of the nodal voltage vector by one. If the same terminal

(3) is employed as the reference for terminal voltages, the new matrix describing the circuit is

```
      □←YR←¯1 ¯1↓Y
 7 ¯12
¯4  25
```

The matrix *YR* is called the <u>reduced nodal conductance matrix</u>. In the expression

```
IR←YR+.×VR                                    (8.12)
```

it relates a <u>reduced</u> vector *IR* of terminal currents

```
IR←I[1],I[2]
```

to a reduced vector *VR* of terminal voltages. The latter vector is not, however, obtained simply by dropping the appropriate element of a vector *V* whose numerical value is known. To take the previous example,

```
      V
2 14 ¯7
```

we must, in shifting the reference point to node 3, add the same amount to each of the elements of *V* to obtain the vector

```
      □←V←V-V[3]
9 21 0
```

from which the third element is now dropped to obtain *VR*:

```
      □←VR←¯1↓V
9 21
```

Calculation of the reduced vector of nodal currents from equation (8.12)

```
      □←IR←YR+.×VR
¯189 489
```

is seen to be in agreement with an earlier calculation.

It may have occurred to the reader that, at some length, we have come to the same conclusion that is drawn in a few lines in Chapter 6, namely that when measuring a 3-terminal component only two currents and two voltages need be considered, and that the component can be described by a 2 by 2 matrix. It should be noted, also, that the vectors *I* and *V* associated with Fig. 8.1 are, in fact, <u>reduced</u> vectors of current and voltage. The names *I* and *V* were used temporarily since the concept of a reduced vector had not been introduced at that point.

We may be tempted to remark that the nodal conductance matrix contains redundancy. But this is not really true. The reduced nodal conductance matrix *YR* only serves as a description of the circuit if we also know which terminal is the reference terminal. In contrast, the nodal conductance matrix *Y* is a complete description. Given *Y*, we may easily generate the reduced nodal conductance matrix for each possible choice of reference terminal:

```
Reference
terminal:      1                    2                    3

          H←0 1 1              H←1 0 1              H←1 1 0

          H/H/[1]Y             H/H/[1]Y             H/H/[1]Y
    ¯25  ¯21                  ¯7   5               ¯7  ¯12
    ¯13   16                  ¯3   16              ¯4   25
```

This procedure is illustrated in Fig. 8.10. Since *Y* constitutes a complete description of the circuit it can be called the Complete Nodal Conductance Matrix of the circuit. It is important to note that in obtaining *YR*, the row and column are deleted from *Y* for different reasons. The column is deleted because a terminal voltage has been set to zero. The row, on the other hand, is deleted not because some quantity is zero but because we do not wish to consider a particular terminal current.

Finally, we remark that although the discussion has employed a circuit as an example, the notion of a complete nodal conductance matrix is equally applicable to a component, a fact which can be explored as an exercise.

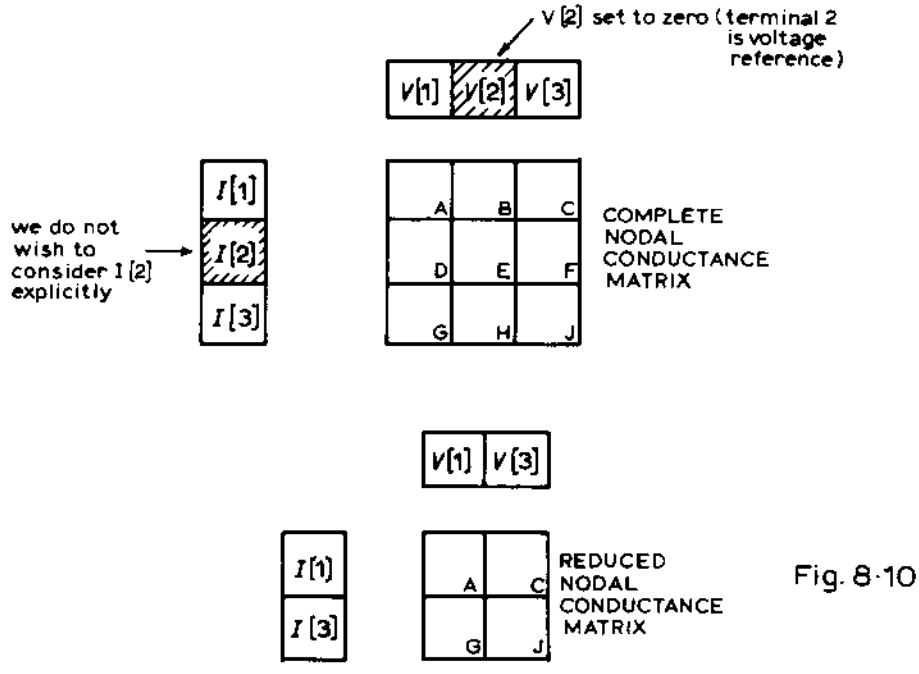


Fig. 8·10

CHAPTER 9

LINEAR CIRCUIT ANALYSIS

In the previous chapter we discovered how to calculate the reduced nodal conductance matrix of a circuit from the properties of its (linear, homogeneous) components and the manner of their interconnection. This matrix, *YR*, relates the reduced vector *IR* of nodal currents to the reduced vector *VR* of nodal voltages according to the nodal equation

```
IR←YR+.×VR
```

(9.1)

If the nodal voltages are known, then the nodal currents can be computed, irrespective of which is considered to be excitation and which is considered to be response.

9.1 THE REDUCED NODAL RESISTANCE MATRIX

It will normally be the case that a circuit must be regarded as being excited by current sources. If the system designer can only connect sources of excitation to a restricted subset of the nodes (i.e., to those nodes which are said to be accessible from outside the circuit), then the current injected into each of the internal nodes must be zero. In other words, for certain nodes we can specify the injected nodal current, whereas we do not know the nodal voltage. Thus, since we must choose, as excitation, either all the nodal voltages (the vector *VR*) or all the nodal currents (the vector *IR*), we make the latter choice. In this case the externally connected sources must be modelled by their Norton equivalent sources, so that current excitation is provided. The circuit requiring analysis is therefore exemplified by Fig. 9.1, where the internal resistance associated with the source of excitation is now regarded as part of the circuit, so that only ideal current sources (many having the value zero) are connected to the circuit nodes. Fig. 9.2, in which all terminals (i.e., the junctions of all components) are explicitly shown, but where the circuit detail is hidden, summarizes the problem to be solved: the excitation vector *IR* and the matrix *YR* in the nodal equation (9.1) are known, and the response vector *VR* must be found.

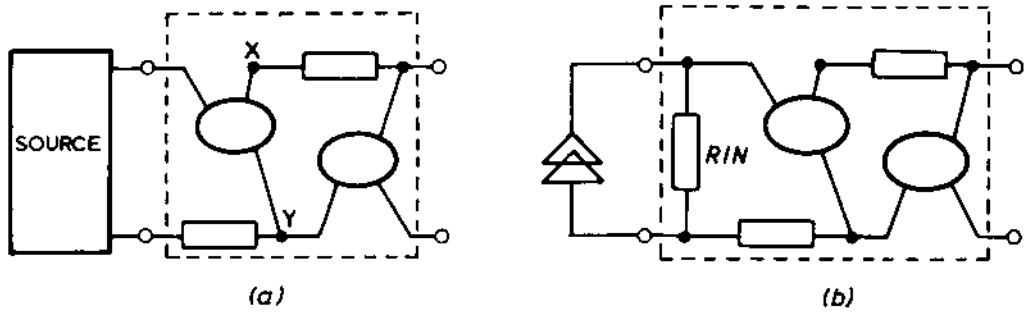


Fig. 9·1 (a) The current injected into nodes X and Y will always be zero, because they are internal nodes.
(b) To associate current excitation with every terminal, the internal resistance of the Norton source model is moved into the circuit.

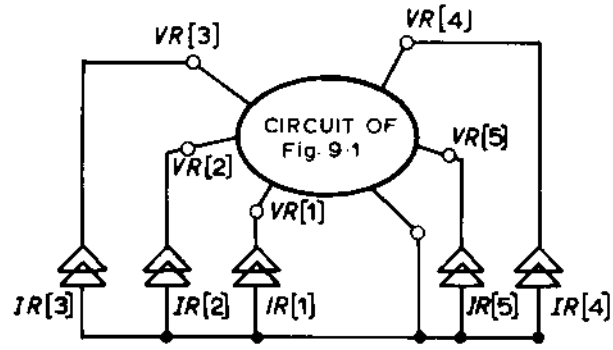


Fig. 9·2 General excitation of a circuit by current sources.

Faced with this problem, it is now possible to appreciate and exploit the similarity between the description of a circuit (9.1)

```
IR←YR+.×VR
```

and the description

```
I←G+.×V
```

of an N-terminal component (see Section 6.8). To obtain the voltage response of a component to current excitation (see Section 6.5) we performed the calculation

```
V←I⌹G                                                    (9.2)
```

A similar calculation yields the voltage response of a circuit to current excitation. Take as an example the circuit of Fig. 8.6, repeated as Fig. 9.3, whose reduced nodal conductance matrix is

```
      YR
 7  ¯12
¯4   25
```

if node 3 is chosen as the reference node. The reduced vector of nodal currents is

```
      IR←¯189 489
```

and therefore (cf. equation 9.2) the voltage response is

```
      □←VR←IR⌹YR                                          (9.3)
9  21
```

Thus, with reference to node 3, the voltages at nodes 1 and 2 are 9 volts and 21 volts respectively, in agreement with a calculation in Section 8.4. It may, however, be necessary to compute the response of the same circuit to different values of current excitation. In this case, rather than proceeding directly to the calculation of (9.3), it is better to consider the relation

```
      VR←ZR+.×IR
```

which is the **inverse** of (9.1), and in which *ZR* is called the **reduced nodal resistance matrix**. Calculation of *ZR* now yields a matrix by means of which the response to any current excitation can be computed without further need to invert *YR*:

```
      □←ZR←⌹YR
   0.197          0.0945
   0.0315         0.0551
```

For the excitation shown in Fig. 9.3, the calculation

```
      □←VR←ZR+.×IR
9 21
```

confirms the earlier result.

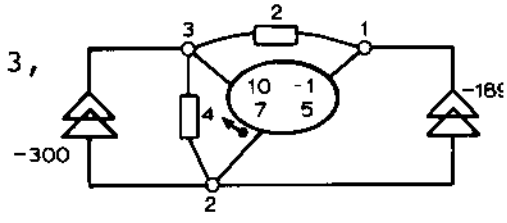


Fig. 9·3

The nodal conductance matrix is a function only of *A* and *YB*. So is *YR* once the reference terminal has been specified, and hence *ZR* also. We therefore define a function *ZEDR* by means of which the reduced nodal resistance matrix of a circuit can easily be calculated

```
    ∇ ZR←A ZEDR YB
[1]   ZR←⌹¯1 ¯1↓A+.×YB+.×⍉A
    ∇
```

This function chooses the reference node as the highest numbered terminal.

9.2 CIRCUIT ANALYSIS

We shall analyse, in some detail, the circuit of Fig. 9.4a. In this figure we have all the information necessary to determine the response of the circuit to the excitation shown.

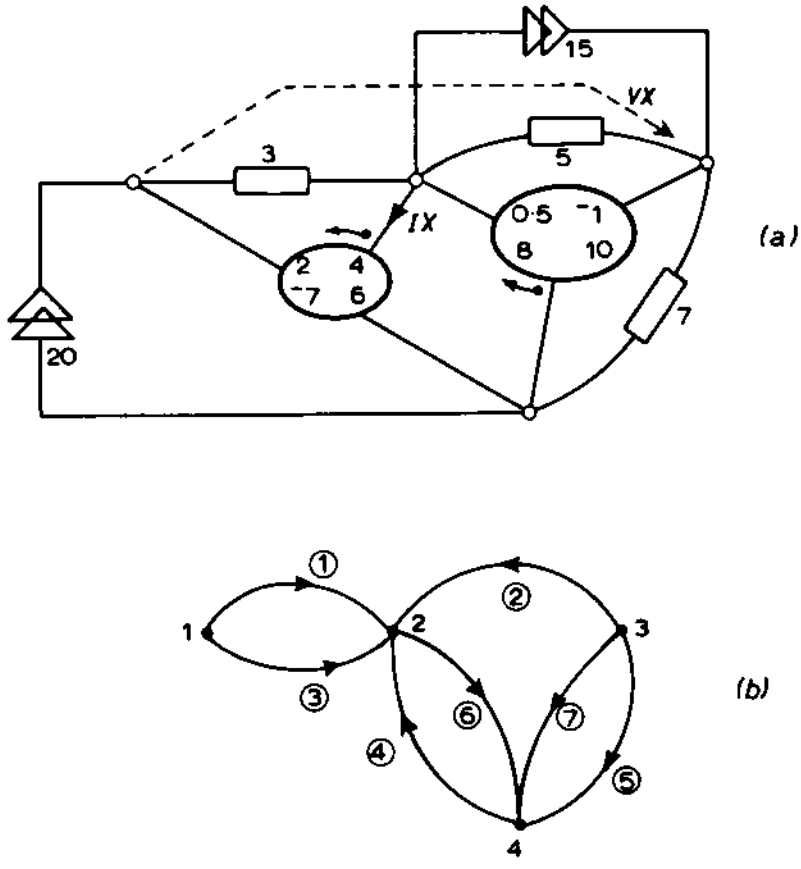


Fig. 9·4 A circuit and its directed graph.

First we draw the directed graph of the circuit (Fig. 9.4b). In representing the two 3-terminal components we take account of their reference terminals. Nodes are numbered arbitrarily. The branches also are numbered arbitrarily, except that consecutive numbers are assigned to the two branches representing a 3-terminal component. Moreover, the order of their numbering must correspond to the order employed in the description of these components, and indicated by the arrow on the dot associated with the component reference node.

Table 9.1 shows the branch connection matrix M, the branch conductance matrix YB, and the incidence matrix A appropriate to the circuit of Fig. 9.4.

```
      M←⍉7 2⍴1 2 3 2 1 2 4 2 3 4 2 4 3 4

      M                                    A←INCID M
1 3 1 4 3 2 3                              A
2 2 2 2 4 4 4                                1  0  1  0  0  0  0
                                           ¯1 ¯1 ¯1 ¯1  0  1  0
                                            0  1  0  0  1  0  1
                                            0  0  0  1 ¯1 ¯1 ¯1

      YB←3 AA 5 AA(2 2⍴2 4 ¯7 6)AA 7 AA (2 2⍴.5 ¯1 8 10)
      YB
3    0    0    0    0    0    0
0    5    0    0    0    0    0
0    0   ¯2    4    0    0    0
0    0   ¯7    6    0    0    0
0    0    0    0    7    0   ¯0
0    0    0    0    0  0.5  ¯1
0    0    0    0    0    8   10
```

Table 9.1

If the highest numbered node (4) is chosen as the reference node, the current excitation of the circuit is described by the reduced nodal current vector *IR*:

```
      IR←20 ¯15 15
```

The function *ZEDR* (which treats the highest numbered node as the reference node) can be used to compute the voltage response:

```
      ⎕←VR←(A ZEDR YB)+.×IR
2.1 ¯1.06 0.826
```

Thus, nodes 1, 2 and 3 have voltages of 2.1, ¯1.06 and 0.826 volts measured with respect to terminal 4.

The voltage vector *VR* may be considered a satisfactory outcome of the circuit analysis. Nevertheless, the circuit designer especially may be interested in knowing the currents and voltages associated with the individual components, and described by the vectors *IB* and *VB*. Recalling the earlier statement of Kirchhoff's voltage law

$$VB = (\lozenge A) +.\times V \qquad (9.4)$$

we extend *VR* by a single zero element (the voltage of the reference terminal, which is the last numbered terminal) to obtain *V*

```
      V←VR,0
```

and use (9.4) to compute the branch voltage vector

```
      ⎕←VB←(⍉A)+.×V
3.16 1.88 3.16 1.06 0.826 ¯1.06 0.826
```

a vector whose dimension is consistent with the number of circuit branches

```
      ⍴VB                              ¯1↑⍴A
7                                  7
```

If a particular voltage of interest - such as *VX* in Fig. 9.4a - does not happen to be associated with a branch, Kirchhoff's voltage law can be invoked to compute it: the subtraction of the appropriate nodal voltages yields:

```
      □←VX←V[3]-V[1]
¯1.27
```

Knowledge of the branch voltage vector allows calculation of the vector of branch currents:

```
      □←IB←YB+.×VB
9.47 9.41 10.5 ¯15.8 5.78 ¯1.35 ¯0.185
```

Again, if a particular component current is not an element of *IB*, it can easily be found by invoking Kirchhoff's current law. For example, the current *IX* in the circuit of Fig. 9.4 is

```
      □←IX←-+/IB[3],IB[4]
5.23
```

As a check on the circuit analysis we have just carried out, we generate the complete nodal current vector *I*:

```
      □←I←IR,-+/IR
20 ¯15 15 ¯20
```

and then compute the nodal current vector that would be predicted from knowledge of the branch currents:

```
      A+.×IB
20 ¯15 15 ¯20
```

They are in agreement.

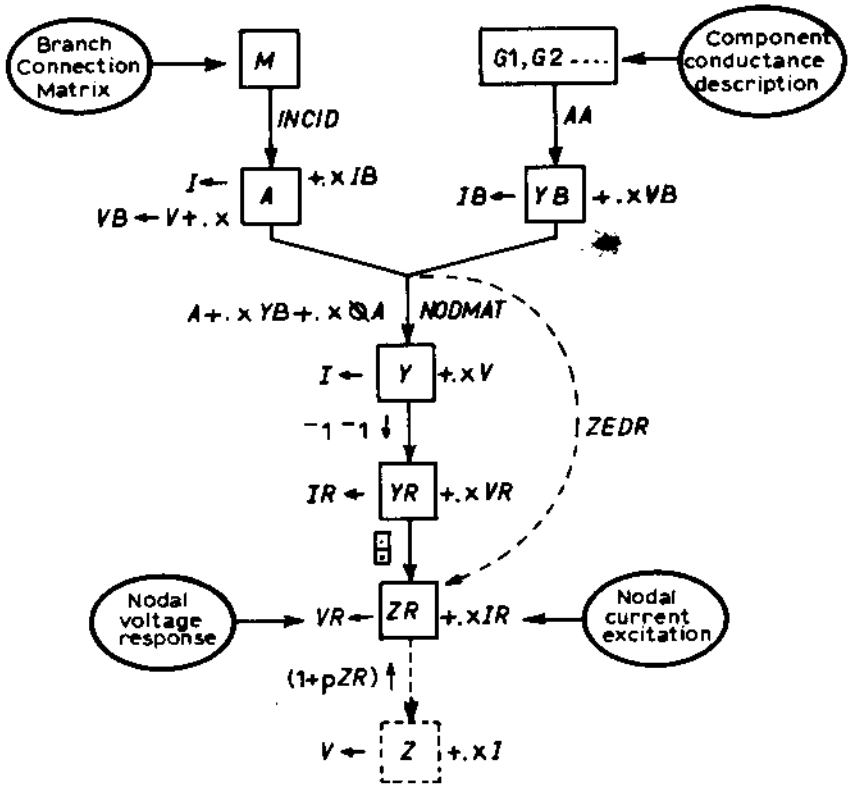
The total power dissipated in the circuit is

```
      □←P←IB+.×VB
70.2
```

and agrees with the calculation of the power supplied by the excitation sources

```
      □←PS←IR+.×VR
70.2
```

It is helpful to summarize, in diagrammatic form, what is involved in linear circuit analysis. This is done in Fig. 9.5.




```
     ∇ A←INCID M;N
[1]    A←((⍳N)∘.=M[1;])-(⍳N←⌈/,M)∘.=M[2;] ∇

     ∇ YB←X AA Y;XX;YY;I
[1]    XX←(¯2↑1 1,ρX)ρX
[2]    YY←(¯2↑1 1,ρY)ρY
[3]    I←(ρXX)+ρYY
[4]    YB←(I↑XX)+(-I)↑YY ∇

     ∇ Y←A NODMAT YB
[1]    Y←A+.×YB+.×⍉A ∇

     ∇ ZR←A ZEDR YB
[1]    ZR←⌹¯1 ¯1↓A+.×YB+.×⍉A ∇
```

Fig. 9·5 The steps involved in linear circuit analysis, and some of the functions involved. (The matrix Z is discussed in section 9·4)

9.3 SYSTEMS OF CIRCUITS

To achieve success in the design of a circuit to perform a given function, the designer will often choose to break the function - and hence the circuit - into two or more constituent parts as shown in Fig. 9.6a. These isolated parts will, in general, be easier to design. At a later stage they will be connected together (Fig. 9.6b) in order to realise the required function. Such an approach is common in the design of a wide range of engineering systems. In the case of electric circuits it poses the question, "given the description of two or more different circuits, how is the description of their interconnection obtained?"

Essentially, the answer is to regard the two constituent circuits as components which are connected together to form a new circuit. In this way the description and analysis of the resulting circuit can be achieved by the method we have developed and with which we are now familiar.

Consider, for example, the circuit of Fig. 9.7 which is formed by the interconnection of the circuit of Fig. 9.4 (here called circuit A) and the circuit of Fig. 8.6 (here called circuit B). The new circuit and its two constituent components are shown in Fig. 9.8. We have already computed the reduced nodal conductance matrix of circuit B, and that for A is readily found:

Circuit *A*

```
      YRA
5     ¯9      0
2     13.5    ¯6
0      3      22
```

Circuit *B*

```
      YRB
 7    ¯12
¯4     25
```

It only remains to draw and describe the directed graph associated with their interconnection (Fig. 9.9)

```
M←⍉5 2 ⍴ 1 5 3 5 4 5 2 4 5 4
```

and to generate the branch conductance matrix describing the two circuits:

```
YB←YRA AA YRB
```

Note, however, that whereas the nodes can be numbered arbitrarily, the branches must be numbered consecutively in the order indicated by the component reference arrows if the matrices *YRA* and *YRB* are to describe the components *A* and *B* correctly.

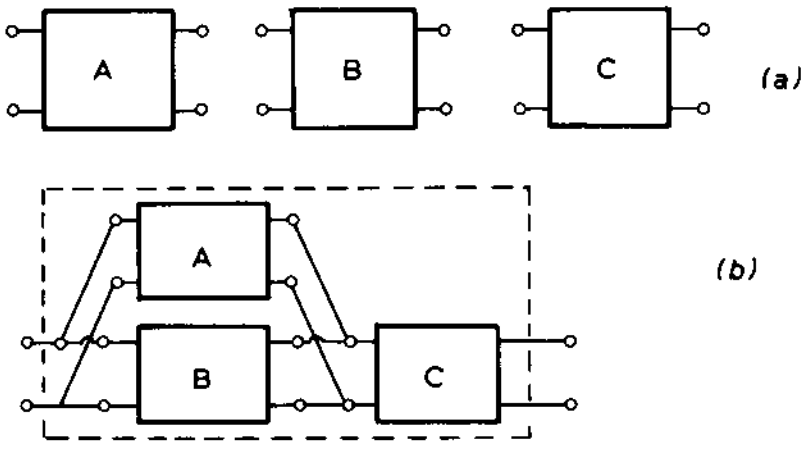


Fig. 9·6 Design of a circuit by breaking down the overall function into parts.

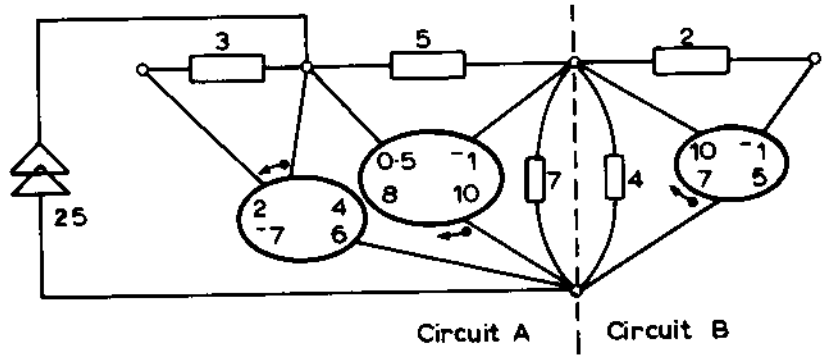


Fig. 9·7 The system to be analysed : the interconnection of circuits A and B.

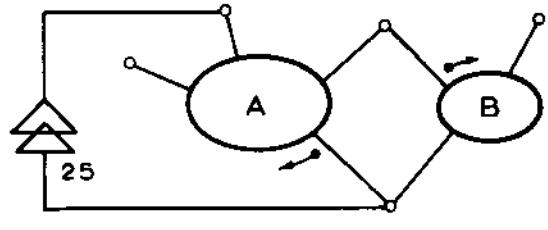


Fig. 9·8 Representation of circuits shown in Fig. 9·7 as two components.

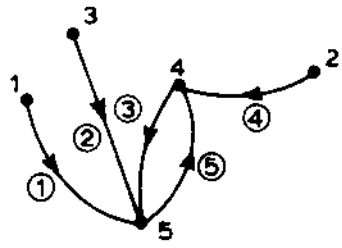


Fig. 9.9 The directed graph of the circuit of Fig. 9.8

In view of the node numbering shown in Fig. 9.9, the excitation of the circuit is described by the reduced nodal current vector

```
      IR←0 0 25 0
```

so that the reduced vector of nodal voltages (i.e., the response) is:

```
      □←VR←((INCID M) ZEDR YB)+.×IR
2.56 0.076 1.42 ¯0.106
```

In other words, with reference to node 5 in Fig. 9.9 (i.e., the lowest terminal in Fig. 9.7) the voltages of nodes 1, 2, 3 and 4 are 2.56 volts, 0.076 volts, 1.42 volts and ¯0.106 volts respectively.

In the above calculation we have made use of the function *ZEDR*. Alternatively we could have computed, in turn, the incidence matrix *A*, the complete nodal conductance matrix *Y*, the reduced nodal conductance matrix *YR* and then, by inversion, the reduced nodal resistance matrix *ZR*; the matrix product with *IR* would then yield the desired voltage response.

What is important to understand is the idea of regarding each circuit as a component: once we have learned how to describe the component that is itself an interconnection of components we are able to handle, with clarity, many problems in the analysis of linear homogeneous circuits.

9.4 THE COMPLETE NODAL RESISTANCE MATRIX

The reader may well comment, at this point, that we have proceeded from a complete nodal conductance matrix *Y* having *N* rows and columns to a reduced matrix *YR* and thence, by inversion, to another reduced matrix, *ZR*, each having *N*-1 rows and columns: is it now possible to proceed

to a resistance matrix, Z, having N rows and columns (see Fig. 9.5), and which in some respect would be more general than ZR? Would such a matrix be useful? It will also have been noticed that the row and column deleted from Y to obtain YR have always been associated with the same node: need this be so? Surely - one might point out - any nodal current can be determined from knowledge of the remaining nodal currents? Let us try to resolve these questions by means of an example.

The circuit of Fig. 8.6, repeated here as Fig. 9.10, was found to be described by the complete nodal conductance matrix Y:

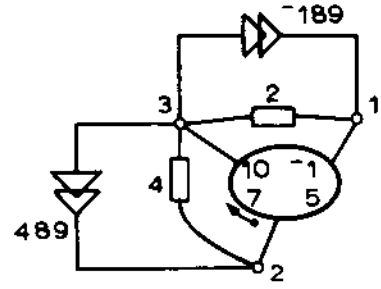


Fig. 9·10

```
      Y
 7  ¯12   ¯5
¯4   25  ¯21
¯3  ¯13   16
```

Choice of terminal 3 as the reference terminal implied that $V[3]\leftarrow 0$ and that we would not explicitly be concerned with the current $I[3]$ entering node 3. This choice allowed the circuit to be described by the relation

```
IR←YR+.×VR
```

where

```
VR←V[1],V[2]
IR←I[1],I[2]
```

and the reduced nodal conductance matrix YR is given by

```
      H←1 1 0
      □←YR←H/H/[1]Y
 7  ¯12
¯4   25
```

Matrix inversion

```
      □←ZR←⌹YR
  0.197          0.0945
  0.0315         0.0551
```

yields the reduced nodal resistance matrix ZR involved in the relation

```
VR←ZR+.×IR
```

The two rows of ZR correspond to the voltage of nodes 1 and 2; there is no third row because the third column of Y was eliminated ($V[3]\leftarrow 0$) by the choice of node 3 as the reference node. However, we may now expand the matrix ZR so that it possesses a third row, consisting entirely of zeros and commensurate with the fact that $0=V[3]$ regardless of the currents that flow:

```
      H\[1]ZR
 0.197          0.0945
 0.0315         0.0551
 0              0
```

Similarly, we may add a third column corresponding to the current associated with the reference node. Since this current is not explicitly involved in the calculation of nodal voltages, the elements of this column must be zero in order to produce a zero result when multiplied by the current. The result

```
      □←Z←H\H\[1]ZR
 0.197          0.0945         0
 0.0315         0.0551         0
 0              0              0
```

is called the <u>complete nodal resistance matrix</u>, and relates the complete vectors of nodal current and voltage (see Fig. 9.5) by the expression

$$V\leftarrow Z+.\times I \qquad (9.4)$$

Whereas use of the reduced matrix ZR demands prior knowledge of which node is the reference node, the matrix Z is complete in the sense that no additional information is needed. As a check we compute, for the excitation shown in Fig. 9.10

```
      I←¯189 489 ¯300
```

the complete vector of nodal voltage response, and note its agreement with an earlier calculation:

```
      V←Z+.×I
9 21 0
```

If the highest numbered node is chosen as the reference node, a simple relation exists between the complete and reduced nodal resistance matrices:

```
      □←Z←(1+ρZR)↑ZR
 0.197          0.0945         0
 0.0315         0.0551         0
 0              0              0
```

It is not, of course, necessary to choose the highest-numbered node as the reference node. If, instead, we choose node 2 by using

```
      H←1 0 1
```

and calculate the new complete nodal resistance matrix,

```
      Z←H\H\[1]⌹H/H/[1]Y
```

the matrix product of Z and the complete nodal current vector

```
      V←Z+.×I
¯12 0 ¯21
```

yields a nodal voltage vector which is in agreement with that computed immediately above - the changes in the elements are due solely to the choice of a different reference node.

Up to now, the same node has been associated with the voltage reference point and with the current that is not explicitly involved in the calculation of response. The fact that two different nodes can be chosen for these two purposes can be demonstrated by example. If node 3 is chosen as the voltage reference node

```
      H←1 1 0
```

and node 2 is chosen as the node whose current is to be ignored

```
      K←1 0 1
```

the same procedure employed above

```
      ⎕←Z←K\H\[1]⌹H/K/[1]Y
 ¯0.102          0          ¯0.0945
¯0.0236          0          ¯0.0551
  0              0           0
```

yields a complete nodal resistance matrix whose matrix product with I still generates the correct result:

```
      ⎕←V←Z+.×I
9 21 0
```

Further exploration of the complete nodal resistance matrix is left as an exercise.

CHAPTER 10

LINEAR CIRCUIT PROPERTIES

For a circuit composed of linear homogeneous components the previous chapter has shown how the complete or reduced nodal resistance matrix can be found, and how it can be used to determine the circuit's response to any set of nodal excitation currents. We are, therefore, now able to explore a number of important properties of these circuits.

10.1 SELF- AND MUTUAL-RESISTANCES

The circuit of Fig. 10.1 is described by its directed graph *M* and its branch conductance matrix *YB*:

```
      □←M←⍉8 2⍴1 2 1 5 1 3 2 3 2 5 3 5 4 5 4 3
 1 1 1 2 2 3 4 4
 2 5 3 3 5 5 5 3
      T←2 2⍴3 ¯2 4 5
      □←YB←2 AA 1 AA 4 AA 7 AA T AA 10 AA 3

   2   0   0   0   0   0   0   0
   0   1   0   0   0   0   0   0
   0   0   4   0   0   0   0   0
   0   0   0   7   0   0   0   0
   0   0   0   0   3  ¯2   0   0
   0   0   0   0   4   5   0   0
   0   0   0   0   0   0  10   0
   0   0   0   0   0   0   0   3
```

By means of previously defined functions, its reduced nodal resistance matrix is easily obtained:

```
      □←ZR←(INCID M) ZEDR YB

0.195      0.0492     0.0668     0.0154
0.0735     0.114      0.0719     0.0166
0.0547     0.0294     0.081      0.0187
0.0126     0.00678    0.0187     0.0812
```

Use of the function *ZEDR* implies that the highest-numbered node (5) is the reference node.

The matrix *ZR* can be used to compute the nodal voltage response to any set of nodal excitation currents. For the excitation *IR*←0 4 ¯4 ¯7 shown in Fig. 10.1 the vector of nodal voltages is

```
      □←VR←ZR+.×IR
¯0.178 0.0505 ¯0.337 ¯0.616
```

Clearly, if a current is injected into every node, and every nodal voltage is of interest, then every element of the matrix *ZR* is involved in the calculation of these voltages. However, if the voltage of only one node (with respect to the reference) is of interest, then only the relevant row of *ZR* need be known. The voltage at node 3 of Fig. 10.1 can, for example, be found from the calcuation

```
      +/ZR[3;]×IR
¯0.337
```

or

```
      ZR[3;]+.×IR
¯0.337
```

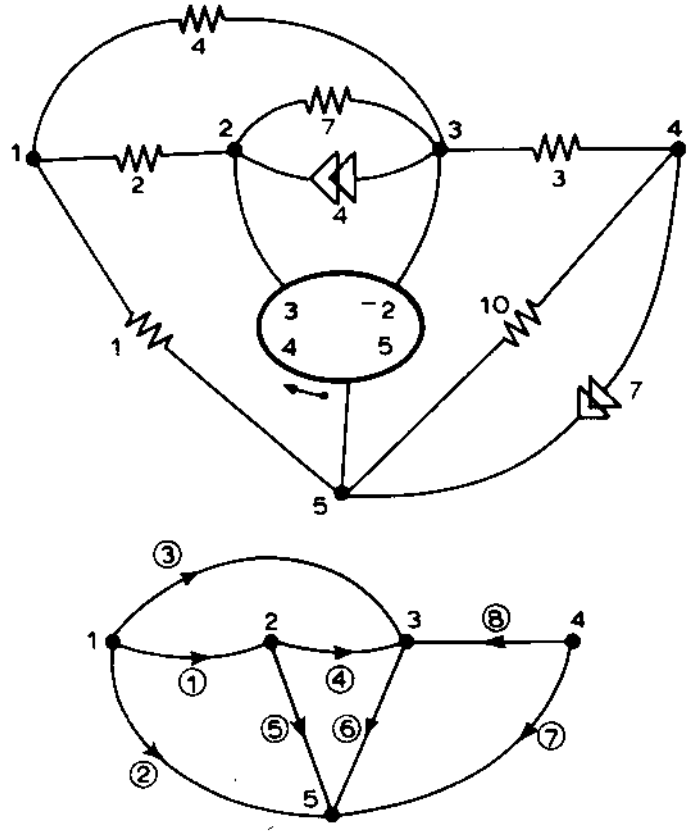


Fig.10-1 A circuit and its directed graph.

instead of by indexing the entire computed vector of nodal voltages:

```
      (ZR+.×IR)[3]
¯0.337
```

Furthermore, if the excitation of the circuit is a single current source with one of its terminals connected to the reference terminal, the calculation of the response voltage at a single node involves only one element of the *ZR* matrix. Thus, if the circuit of Fig. 10.1 is now excited in the manner shown in Fig. 10.2a, the calculation of the voltage at node 3 with respect to the reference terminal (5) involves only a scalar product:

```
      ⎕←VR[3]←ZR[3;2]×40
1.17
```

In this way it becomes evident that each element of *ZR* represents the voltage response, at the node identified by the row index, to current excitation at the node identified by the column index. Thus,

```
      ZR[4;2]
0.00678
```

is the voltage response at node 4 (measured with respect to the reference node) due to a current of 1 ampere injected into node 2, where the return path for the current is via the reference node. Indexing an entire column of *ZR*, say

```
      ZR[;2]
0.0492 0.114 0.0294 0.00678
```

yields the voltages at all the nodes due to a current of 1 ampere injected at the node (here 2) associated with the column of *ZR*.

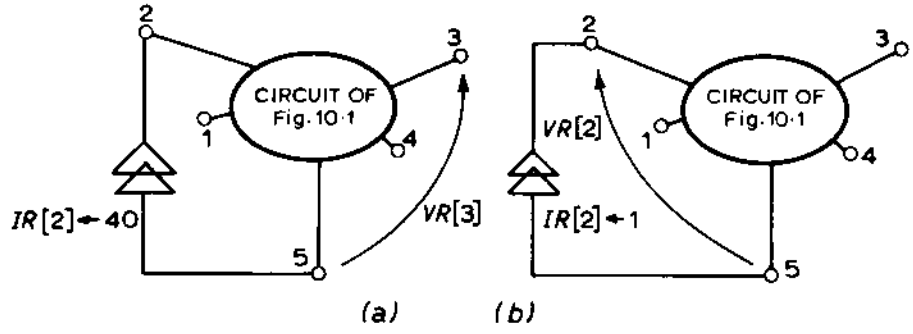


Fig.10·2 Excitation of the circuit of Fig. 10·1 by a single current source.

If the row and column indices differ, indicating that the response and excitation are associated with different nodes, the element of *ZR* is called a **mutual resistance**. Thus,

```
      ZR[1;3]
0.0668
```

is the mutual resistance *VR*[1]÷*IR*[3] between node 3 (where the excitation is applied) and node 1 (where the response is measured). It follows that all off-diagonal elements of the reduced nodal resistance matrix are mutual resistances. The **diagonal** elements of *ZR*, on the other hand, relate the voltage at a node to the current excitation at the same node, and for this reason are called self resistances. Thus, in Fig. 10.2b, the injection of a current of 1 ampere into node 2 results in a voltage at node 2 that can be determined simply from knowledge of the self resistance of this node:

```
      □←VR[2]←ZR[2;2]×1
0.114
```

10.2 THE SUPERPOSITION PRINCIPLE

The relation between nodal response voltages and nodal excitation currents

```
VR←ZR+.×IR
```

possesses two important characteristics: linearity and homogeneity. These properties arise, in turn, from the linearity and homogeneity of the branch relations describing the components which together make up the circuit. An important consequence of these properties, and one which has previously been encountered in Section 5.2, is that the response to a number of excitation currents is equal to the sum of the responses due to each excitation current considered separately. This property is of such immense importance that it is given a name:

The Superposition Principle

> The response of a circuit, composed of linear homogeneous resistive components, to the simultaneous application of a number of nodal excitation currents, is the algebraic sum of the responses due to each excitation current acting alone.

A numerical example can be taken from Fig. 10.1, for which the voltage response *VR* to three excitation currents acting simultaneously has been calculated:

```
      □←VR←ZR+.×IR
¯0.178 0.0505 ¯0.337 ¯0.616
```

The same result is obtained by computing the voltage response of the circuit to separate excitation currents described by the vectors

```
IRA←0 4 0 0
IRB←0 0 ¯4 0
IRC←0 0 0 ¯7
```

and then adding the resulting responses:

```
      □←VR←(ZR+.×IRA)+(ZR+.×IRB)+(ZR+.×IRC)
¯0.178 0.0505 ¯0.337 ¯0.616
```

By observing that *IR←IRA+IRB+IRC* it is seen that the agreement in the calculated *VR* is a consequence of the distributivity of the matrix product over addition:

$$(ZR+.\times IR)=(ZR+.\times IRA+IRB+IRC)=(ZR+.\times IRA)+(ZR+.\times IRB)+(ZR+.\times IRC)$$

That the components must be linear and homogeneous for the superposition principle to be valid will be emphasized during the discussion of nonlinear circuits in Chapter 11.

It will often be found convenient to invoke the superposition principle in discussions of circuit properties, as the following example will show. In the treatment of mutual resistances it was assumed that the return path for the single excitation current was the reference node, and that the voltage of interest was measured between the node in question and the reference node. Thus, for the excitation shown in Fig. 10.3, the ratio of $\underline{VR}$ and $\underline{IR}\leftarrow 20$ is *ZR*[3;2] and $\underline{VR}$ is easily computed:

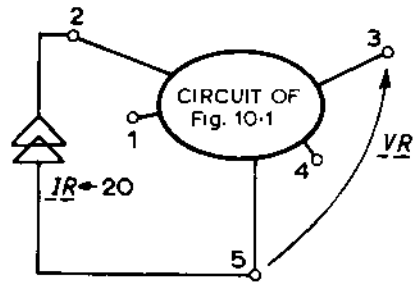


Fig. 10-3

```
      □←VR←ZR[3;2]×20
0.587
```

However, if the return path for the excitation current, as in Fig. 10.4a, is via a node other than the reference node, then the voltage response $\underline{VR}$ is given by

```
      □←VR←ZR[3;]+.×¯20 20 0 0
¯0.506
```

Alternatively, we can say that $\underline{VR}$ is the sum of two components, one

```
      □←VRA←ZR[3;2]×20
0.587
```

due to the current injected at node 2, and the other

```
      □←VRB←ZR[3;1]×¯20
¯1.09
```

due to the current injected at node 1; their sum

```
      □←VR←+/VRA,VRB
¯0.506
```

yields the correct value for the voltage at node 3. It is useful to note that exactly the same calculation is appropriate if the excitation is as shown in Fig. 10.4b. This emphasizes the fact that the two sets of excitation are equivalent; the net current entering node 5 in Fig. 10.4b is zero, so this node is effectively disconnected, as in Fig. 10.4a. Thus, we see that if the return path for excitation current is not the reference node, then the calculation of the voltage at a single node requires knowledge of <u>two</u> elements of the reduced nodal resistance matrix. If, in addition, the voltage to be computed is measured between two nodes, <u>neither</u> of which is the reference node, then <u>four</u> elements of the reduced nodal resistance matrix are involved.

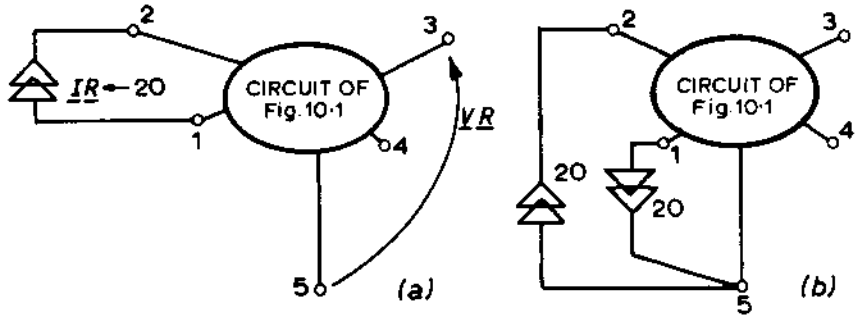


Fig.10·4

10.3 EQUIVALENT CIRCUITS

In many cases the <u>user</u> of a circuit (e.g., the system designer: see Chapter 1) is not primarily interested in how it is constructed, but rather in the <u>overall</u> function it performs. The circuit of Fig. 10.5a, for example, may be delivered by its designer to its user in a box (dotted line) from which only two terminals protrude, in view of the user's intention merely to connect a resistance RL between these two terminals.

If the user wishes to predict the current that will flow in *RL* as this resistance is varied over a specified range, it would appear that he would have to discover what circuit lies within the box, and then perform an analysis of the entire circuit (box connected to *RL*) for each value of *RL*. We shall see, however, that the tedium of such a series of calculations can be reduced considerably, and with the benefit of additional insight into the properties of the circuit, by judicious application of the superposition principle.

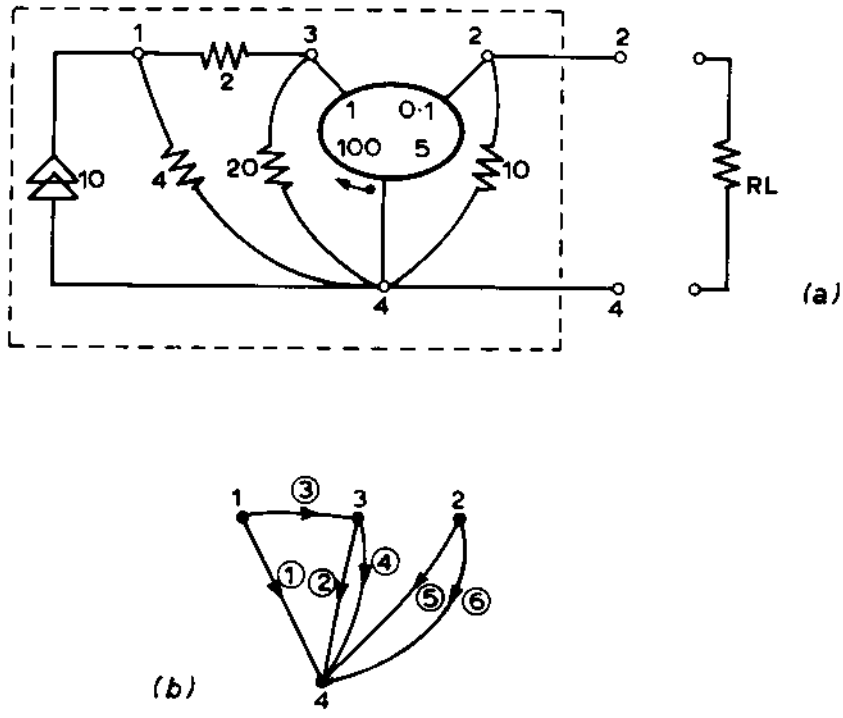


Fig. 10·5

In computing the reduced nodal resistance matrix of the circuit within the box in Fig. 10.5a (i.e., with *RL* neglected) we make the assumption that one of the two terminals to which the resistance *RL* is connected is the reference for nodal voltages: here we have chosen terminal 4:

```
      M←⍉6 2⍴1 4 3 4 1 3 3 4 2 4 2 4
      YB←4 AA 20 AA 2 AA (2 2⍴1 .1 100 5) AA 10
      ⎕←ZR←(INCID M) ZEDR YB
 0.172          ¯0.000103       ¯0.015
¯0.103           0.0687         ¯0.308
 0.0154         ¯0.000308        0.046
```

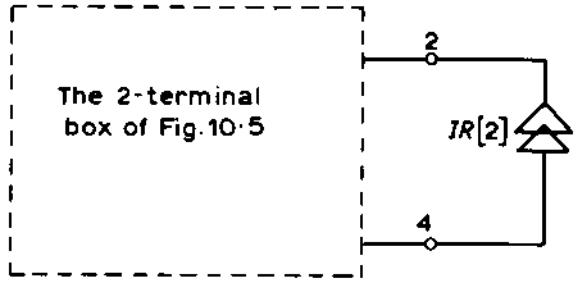


Fig.10·6 The two-terminal box of Fig.10·5 excited by a current source simulating the current in RL

For the circuit exclusive of RL, the nodal current excitation is described by the vector $IR\leftarrow 10\ 0\ 0$. Now consider the connection of RL between nodes 2 and 4. By invoking the substitution theorem (discussed in an exercise) we may replace RL (Fig. 10.6) with a current source whose value $IR[2]$ is the current in RL, without disturbing the circuit. The resulting expression for the voltage $VR[2]$ between the exposed terminals 2 and 4 now contains a term associated with this source:

```
VR[2]←(¯0.103×IR[1])+(0.0687×IR[2])+(¯0.308×IR[3])
```

What is the physical interpretation of the terms in this expression? The sum of the first and third terms yields the value that $VR[2]$ assumes if $IR[2]$ is zero:

```
      □←VOC←VR[2]←+/ZR[2;1 3]×IR[1 3]
¯1.03
```

It is consequently called the open-circuit voltage VOC (between the exposed terminals 2 and 4). The second term $(0.0687\times IR[2])$ in the expression for $V[2]$ can be written as $R\times IR[2]$, where $R\leftarrow ZR[2;2]$.

The expression for $VR[2]$ can therefore be rewritten as

```
VR[2]←VOC+R×IR[2]
```

But this expression also relates the voltage $VR[2]$ and current $IR[2]$ in the circuit of Fig. 10.7. In other words, as far as the relation between $VR[2]$ and $IR[2]$ is concerned, there is no difference between the box of Fig. 10.5 and the box of Fig. 10.7. Put another way, there is no way of making measurements between terminals 2 and 4 that will distinguish between the two boxes. For this reason the two (2-terminal) boxes are said to be <u>electrically equivalent</u>. In view of the ease with which the simple circuit of Fig. 10.7 can be handled - both computationally and mentally - this equivalence is of considerable importance. Since it is identical in form and purpose to the Thevenin model of a

battery first encountered in Chapter 4, it is called the Thevenin Equivalent Circuit of the circuit of Fig. 10.5. We may conclude that instead of connecting the resistance *RL* between terminals 2 and 4 of Fig. 10.5 and analysing the whole circuit for each different value of *RL*, we may now address ourselves to the far simpler task of analyzing the circuit of Fig. 10.8 for each different value of *RL*, an enormous saving of effort.

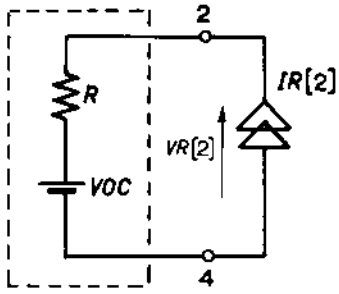


Fig.10·7
The Thevenin equivalent of a 2-terminal box, such as the one shown in Fig.10·5

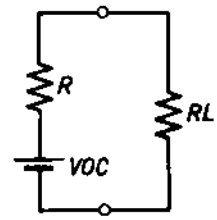


Fig. 10·8
Use of a Thevenin equivalent circuit to compute the current in RL when connected to the 2-terminal box of Fig.10·5

It is important to remember that only two measurements need be carried out to find the Thevenin equivalent of a two-terminal circuit. First, the two terminals are left on open circuit and the open-circuit voltage *VOC* measured. Computationally, the determination of *VOC* first involves identifying the node *T*

```
      T←2
```

which, together with the (highest-numbered) reference node

```
      N←4
```

defines the two available terminals. The vector of internal nodes

```
      □←INTNODES←(~(⍳N)∈T,N)/⍳N
1   3
```

then allows the open-circuit voltage to be computed:

```
      □←VOC←VR[T]←+/ZR[T;INTNODES]×IR[INTNODES]
¯1.03
```

The second parameter to be determined is the self-resistance of the available node, which is the value of the internal resistance of the Thevenin equivalent circuit:

```
      □←R←ZR[T;T]
0.0687
```

The conditions under which the internal resistance is measured in the laboratory can be deduced by interpreting the term $ZR[T;T]$ of the reduced nodal resistance matrix: it is the ratio of the voltage at node T and the current excitation applied to the same node when all other nodal excitation currents are set to zero (Fig. 10.9).

A two-terminal circuit can also be represented by a Norton Equivalent circuit (Fig. 10.10), whose derivation from the Thevenin equivalent should be familiar from an earlier exercise. The current source has a value equal to the current that flows through a short circuit connected between the two accessible terminals, and is equal to $VOC \div R$ (see Fig. 10.8). The resistance R is identical with the internal resistance of the Thevenin equivalent.

Exploitation of the Thevenin equivalent of a complex circuit is illustrated in the following section.

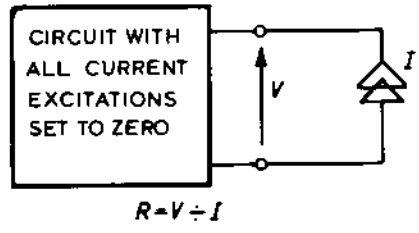


Fig.10·9
Measurement of the Thevenin resistance R of a 2-terminal box.

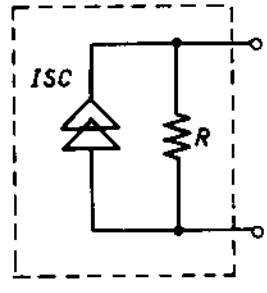


Fig.10·10 The Norton equivalent of a 2-terminal box, where ISC is the current that flows in a short circuit connected between its terminals.

10.4 MAXIMUM AVAILABLE POWER

Economy in the analysis of the circuit of Fig. 10.5 was the original motivation for obtaining the Thevenin model of Fig. 10.7. Nevertheless, by recalling the discussion of Thevenin models presented in Chapter 4, additional insight into the circuit of Fig. 10.5 can be obtained with

negligible additional calculation. It was shown in Chapter 4 that the power supplied to a resistance connected to a Thevenin model is a maximum when that resistance is equal to the internal resistance of the model. We can therefore state, without further reference to the circuit of Fig. 10.5, that the maximum power is delivered to *RL* if

```
RL←0.0687
```

Thus, the task of finding the value of *RL* associated with the supply of maximum power to *RL* in Fig. 10.5 is equivalent to that of finding the internal resistance of the Thevenin model of the (2-terminal) circuit to which *RL* is connected. In this calculation the value of the other Thevenin parameter - the open-circuit-voltage *VOC* - is irrelevant. When the supply of power to *RL* is maximized by the proper choice of *RL*, the two-terminal circuit and the resistance *RL* are said to be matched. It is not surprising that this term is frequently encountered, since it is often of crucial importance to extract the maximum possible power from a circuit or device.

When the condition for maximum power transfer (from the box in Fig. 10.5 to the resistance *RL*) is established by proper choice of *RL*←0.0687, calculation of the actual value of the power supplied to *RL* now requires additional knowledge of the value of *VOC* (¯1.03):

```
      □←MAXP←RL×(VOC÷2×RL)*2
3.83
```

However, the maximum power can profitably be expressed in terms of the internal resistance *R* of the Thevenin model (which equals *RL*):

```
      □←MAXP←R×(VOC÷2×R)*2
3.83
```

since it is now apparent that the maximum power is determined solely by the Thevenin model. Thus, if both parameters of a Thevenin model are known, a value for the maximum available power from the model can be computed, and is independent of any external connection. In other words, a value for the maximum available power can be assigned to a Thevenin model (Fig. 10.11) without regard to the manner in which it is connected to an external circuit. Certainly, the external connection (*RL* in Fig. 10.8) determines the actual power delivered, but the maximum that is available is determined exclusively by the model.

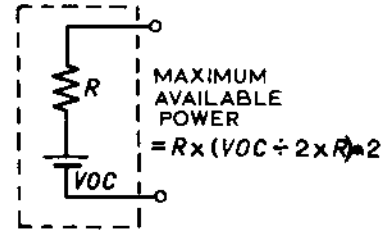


Fig. 10.11

10.5 CONSERVATION OF ENERGY

For the circuit of Fig. 10.12 it is easy to calculate the current I flowing between the two 2-terminal components A and B and hence the voltage V which is common to these components:

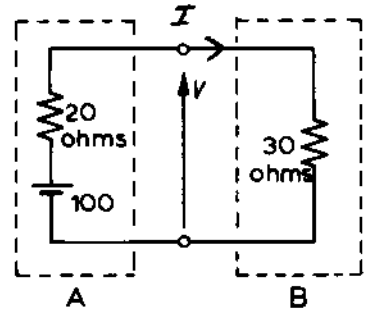


Fig.10·12

```
      □←I←100÷20+30
2
      □←V←30×I
60
```

Recalling the discussion of power and energy in Chapter 4, we compute the power supplied to components A and B:

```
      □←PA←60×¯2                    □←PB←60×2
¯120                              120
```

Thus in T seconds the total energy supplied to the single resistor B and dissipated as heat is

```
      EB←120×T
```

The fact that the rate of energy supply (PA) to component A is negative means that component A is acting as a supply of energy; in T seconds it supplies an energy $EA←120×T$. For this simple example we note that energy is conserved, since the rate ($-PA$) at which power is supplied by component A is equal to the rate (PB) at which it is dissipated by component B. In other words, conservation of energy is equivalent to the condition $0=P←+/PA,PB$ on the total power P supplied to all components.

Is the result we have obtained simply fortuitous? Does the principle of the conservation of energy apply to circuits of any size? Does it depend upon the nature (such as the linearity) of the components?

In the 6-branch circuit of Fig. 10.13, described by its directed graph M and vector of branch conductances G:

```
      M                           G←4 1 8 3 10 7
1 2 2 4 4 3
2 4 3 1 3 1
```

a nodal current excitation IR gives rise to the following nodal voltage response:

```
      IR←7 5 ¯5
      □←VR←((INCID M) ZEDR COND G)+.×IR
0.918 0.875 .0337
```

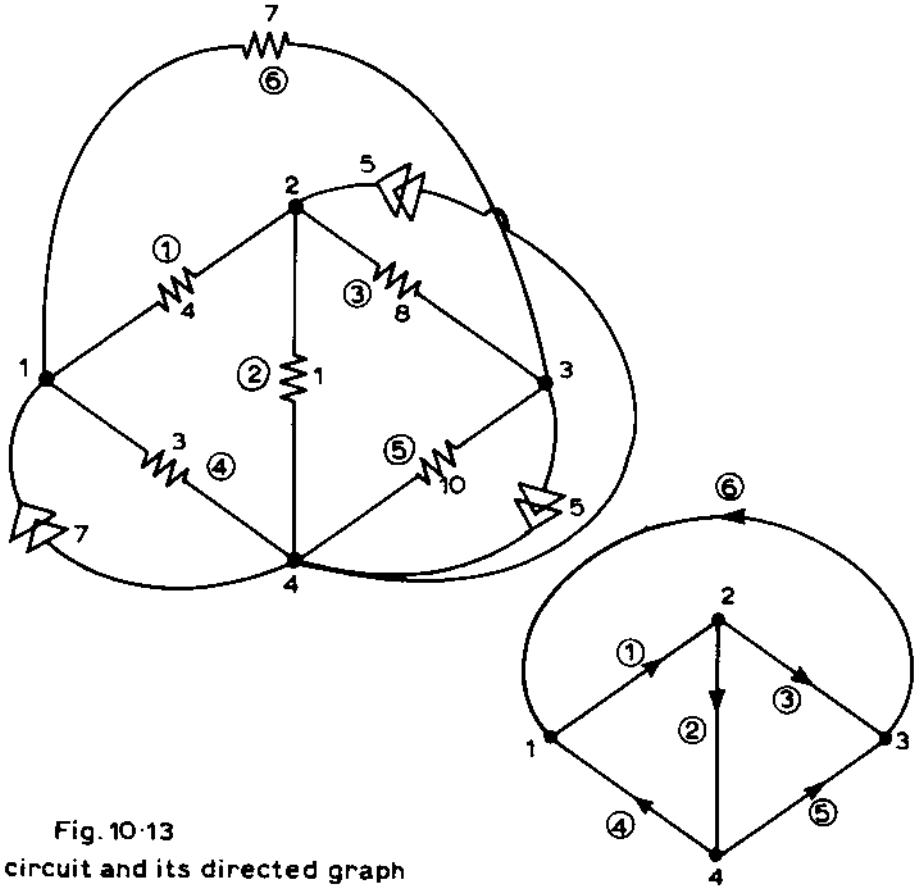


Fig. 10·13
A circuit and its directed graph

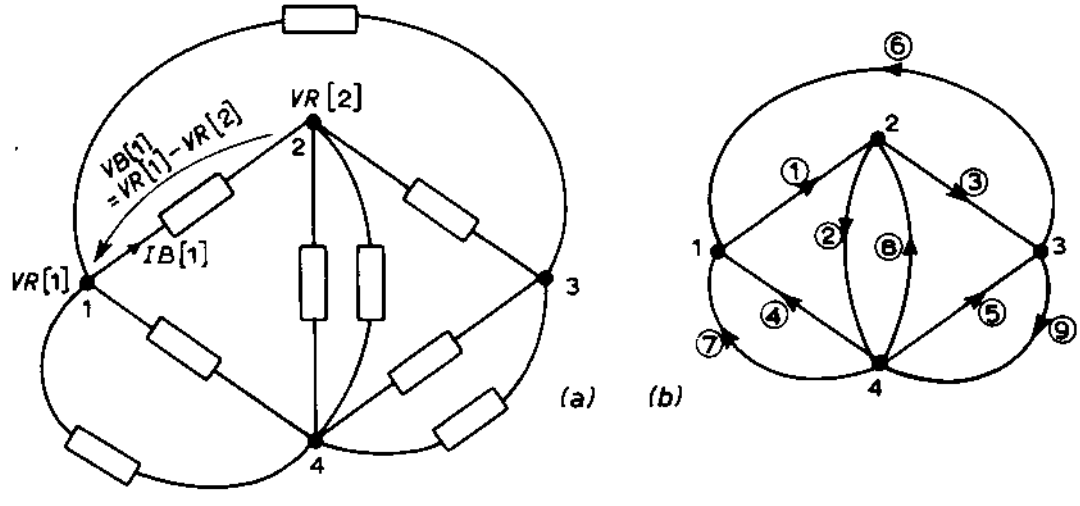


Fig. 10·14
The circuit of Fig. 10·13, with each 2-terminal conductance and current source represented by a 2-terminal box, and the corresponding directed graph.

The vectors of branch voltage and branch current are easily found:

```
      □←VB←(⍉A)+.×VR,0
0.0438 0.875 0.538 ¯0.918 ¯0.337 ¯0.581

      □←IB←G×VB
0.175 0.875 4.30 ¯2.76 ¯3.37 ¯4.07
```

The total power supplied to the branches is positive:

```
      □←P←+/IB×VB
9.12
```

This is expected from the nature of the branches; they are linear homogeneous resistances, so that energy is dissipated as heat for any non-zero current of whatever direction. Since each excitation source has one terminal connected to the reference node, the power supplied to these excitation sources is

```
      □←PS←+/-IR×VR
¯9.12
```

which is seen to satisfy the condition

```
      0=+/P,PS
```

for the conservation of energy. In other words, the excitation sources do not accept energy (*PS* is negative) but supply it, at a rate exactly equal to that at which it is dissipated by the resistances.

Nevertheless, we still lack a fundamental proof of the energy conservation principle. Perhaps surprisingly, such a proof is very simple. The circuit of Fig. 10.13 has been redrawn as Fig. 10.14a without any detailed reference being made to the functional relations of the components and the values of the excitation sources: each is represented by a two-terminal box. The corresponding directed graph is shown in Fig. 10.14b.

Since the vectors *VB* and *IB* now encompass every 2-terminal box in Fig. 10.14, the total power supplied, and which we expect to be zero, is

```
      P←+/PB←VB×IB
```

Kirchhoff's voltage law is invoked to express the power supplied to each box in terms of the two nodal voltages associated with it. Thus, for box 1, the power *PB*[1] supplied to this box can be written as

```
PB[1]
VB[1]×IB[1]
IB[1]×(VR[1]-VR[2])
(IB[1]×VR[1])-(IB[1]×VR[2])
```

and a similar expression can be written for every other box in this circuit. Now let us collect the terms according to the node with which they are associated: the contribution of node 1 due to box 1 is, for example, *IB*[1]×*VR*[1]:

Node	Contribution to *P*
1	*VR*[1]×*IB*[1]+(-*IB*[4])+(-*IB*[6])+(-*IB*[7])
2	*VR*[2]×(-*IB*[1])+(*IB*[2]+*IB*[3]+(-*IB*[8])
3	*VR*[3]×(-*IB*[3])+(-*IB*[5])+*IB*[6]+*IB*[9]
4	*VR*[4]×(-*IB*[2])+*IB*[4]+*IB*[5]+*IB*[7]+*IB*[8]+(-*IB*[9])

The sum of all the contributions in the above table is the total power supplied to the circuit of Fig. 10.4. However, interpretation of this table is considerably eased if the entries are expressed as shown below:

Node	Contribution to *P*
1	*VR*[1]×+/(1 0 0 ¯1 0 ¯1 ¯1 0 0)×*IB*
2	*VR*[2]×+/(¯1 1 1 0 0 0 0 ¯1 0)×*IB*
3	*VR*[3]×+/(0 0 ¯1 0 ¯1 1 0 0 1)×*IB*
4	*VR*[4]×+/(0 ¯1 0 1 1 0 1 1 ¯1)×*IB*

What is now placed in evidence is the fact that the contribution associated with each node involves its incidence vector. Thus, *VR*[2] is multiplied by +/*A*[2;]×*IB*, where *A* is the incidence matrix of the circuit and *IB* is the vector of branch currents. By Kirchhoff's current law +/*A*[2;]×*IB* is zero, and hence its product with *VR*[2] is also zero. Since the same result holds for each node, the total power *P*←+/*VR*×*A*+.×*IB* supplied to the circuit of Fig. 10.4a is zero. In other words, energy is conserved.

It is very important to note that in the above example (which can clearly be generalized to include any circuit) we have only invoked the two Kirchhoff laws: no reference whatever has been made to the branch relation of each box. We may conclude, therefore, that the principle of the conservation of energy is a consequence only of the two Kirchhoff laws, and that the branch relations are immaterial. Thus, the principle applies to nonlinear as well as linear circuits.

CHAPTER 11

NONLINEAR CIRCUIT ANALYSIS

We have previously encountered circuits containing a nonlinear component. For one example it was found that the inverse relation could be expressed algebraically. Unfortunately the number of components for which this is true is negligibly small. For another circuit, the fact that only one component was nonlinear allowed us to explore a range of situations in order to discover the actual currents and voltages of the circuit. Again, this approach is of limited value, since very few circuits contain a single nonlinear component.

Most practical circuits contain a large number of nonlinear components and can only conveniently be analyzed numerically, using a systematic trial-and-error procedure. In this procedure, a guess is first made at the actual circuit response. If this guess is not compatible with the known excitation of the circuit, a measure of the error forms the basis of an improved estimate of the circuit response. This procedure is repeated as often as necessary and is terminated when, to within an acceptable error, the estimated response and known excitation are compatible. Such a method of analysis, where the same calculation is performed repeatedly, is called an iterative method.

11.1 ITERATIVE ANALYSIS OF A NONLINEAR COMPONENT

The ideas and procedures associated with the iterative analysis of a nonlinear circuit will first be demonstrated by considering the simpler situation of a single nonlinear component. Having mastered this simple case the understanding of nonlinear circuit analysis is considerably eased by viewing the nonlinear circuit as a component, a strategy with which we are already familiar.

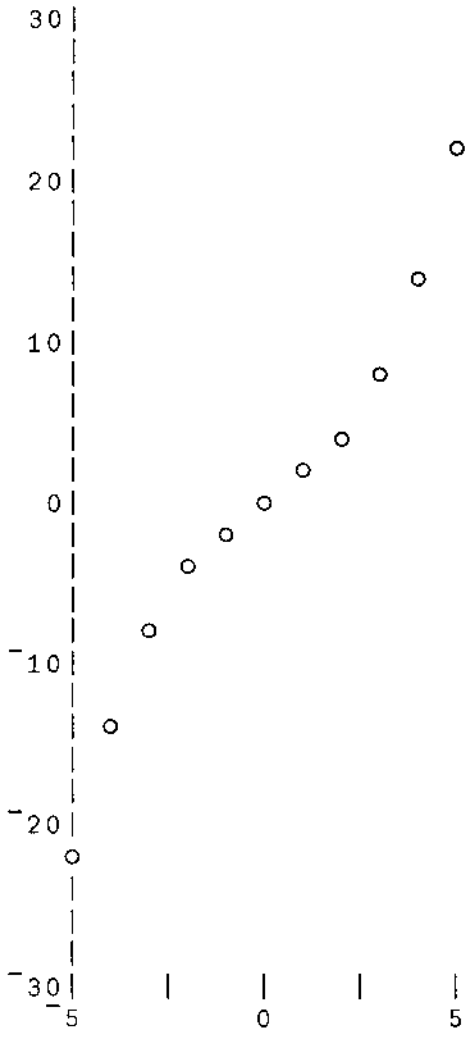
Consider a 2-terminal component whose voltage-current relation is defined by a nonlinear function *COM* which expresses the current as a function of the voltage:

```
     ∇ IR←COM VR
[1]    IR←(2×VR)+0.1×VR*3
     ∇
```

By considering a range of voltage excitation, a plot of current versus voltage is easily obtained:

```
      V←¯6+ι11
      I←COM V

30 30 PLOT I VS V
```



The situation to be considered is shown in Fig. 11.1. The component is excited by an ideal current source *IRX* and we wish to find the voltage response *VR*. It is not possible to obtain the inverse of the function *COM* - that is, an expression for *VR* as a function of *IR* - and we therefore employ an iterative procedure to obtain a numerical result.*

---

* The names *IR* and *VR* are used here in preference to *I* and *V* because we shall shortly be extending our discussion to multiterminal circuits.

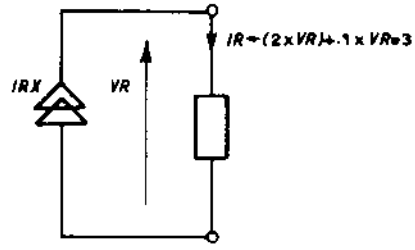


Fig. 11·1
Excitation of a 2-terminal nonlinear component.

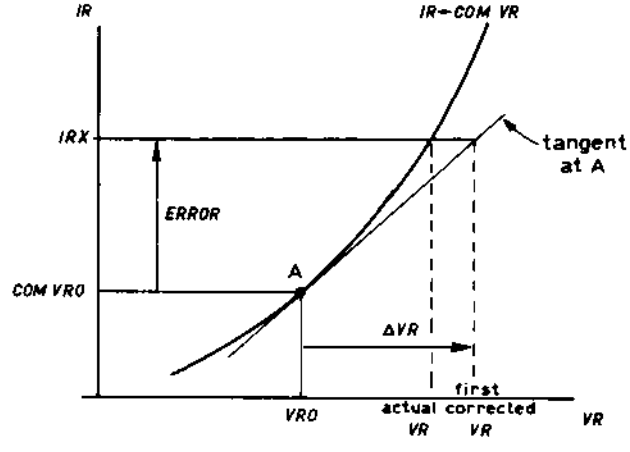


Fig. 11·2
Basis of the Newton-Raphson method for the iterative analysis of a nonlinear circuit.

One possible procedure is embodied in the function *ITER*:

```
    ∇ VR←IRX ITER VR0;ERROR;SLOPE;ΔVR
[1]   VR←VR0
[2]   →L1×1E¯6<|ERROR←IRX-COM VR
[3]  L1:SLOPE←DIFCOM VR
[4]   VR←VR+ΔVR←ERROR÷SLOPE
[5]   →2
    ∇
```

This function takes the value of the excitation current source as its left argument, and our guess at the value of *VR* as its right argument. In explaining the procedure we refer to the diagram of Fig. 11.2, which is based on a plot of the function *COM*:

Line [1] The result *VR* is set initially to the guessed value *VR0*.

Line [2] The component current (*COM VR*) associated with the voltage *VR* is computed by means of the function *COM*. The difference between this current and *IRX* is the error (*ERROR*) in the application of Kirchhoff's current law at one of the component terminals. If the magnitude of the error is less than 1 microampere (other values can be chosen) no further calculation is required and the current value of *VR* is taken as the actual value of the component voltage.

Line [3] If the magnitude of *ERROR* is too large, a correction must be applied to *VR*. To enable the correction to be found, the slope of the function *COM* for the current value of *VR* is calculated by means of the function *DIFCOM*:

```
    ∇S←DIFCOM VR
[1]  S←2+.1×3×VR*2
    ∇
```

Line [4] Examination of Fig. 11.2 shows that for small values of *ERROR*, the necessary correction to *VR* is, to a good approximation,

$$\Delta VR \leftarrow ERROR \div SLOPE$$

Nevertheless, whatever the magnitude of *ERROR*, the correction ΔVR is calculated by this relation, and the corrected value of *VR* computed.

Line [5] A return to line 2 to test the new value of *VR* and repeat the process if necessary.

Let us choose, arbitrarily, a value of 23 amperes for the excitation current and 3000 volts for the initial guess at the value of *VR*. Use of the function *ITER* returns a result for *VR* whose validity is easily checked:

```
      23 ITER ¯3000
5.052199846
      □←I←FUNC 5.052199846
23
```

The progress of the iteration can be observed by putting a trace on line 4, whose result is the corrected value of *VR*:

```
      TΔITER←4
      23 ITER 12

ITER[4]  8.154867257
ITER[4]  5.98903992
ITER[4]  5.169333663
ITER[4]  5.054308024
ITER[4]  5.052200544
ITER[4]  5.052199846
ITER[4]  5.052199846
ITER[4]  5.052199846
5.052199846
```

As well as obtaining experience of the manner in which the algorithm converges upon the solution (i.e., quite rapidly), we note that the same value as before is obtained for *VR* showing, at least for the two examples considered, that the final solution is independent of the initial guess *VR0*. The procedure embodied in the function *ITER* is called the **Newton-Raphson** procedure.

There is a tendency, when examining a method of circuit analysis developed especially for nonlinear circuits, to assume that the method is valid **only** for nonlinear circuits. Though the method is rarely employed for linear circuit analysis, the reader should verify that such use of the function *ITER* is possible.

Rather than determine the response voltage *VR* of the circuit of Fig. 11.1 for a **single** value of the excitation current, it may be necessary to predict *VR* for a sequence of excitation currents. *IRX*, *VR* and *VR0* are now vectors, as are *ERROR*, *SLOPE* and Δ*V*. A check of the functions *COM* and *DIFCOM* shows that they will return the correct vector results if supplied with vector arguments. In fact, to obtain the new function *ITERVEC*, the **only** necessary modification to the function *ITER* ensures, in line 2, that the **maximum reduction** of the magnitude of *ERROR* is tested :

```
     ∇ VR←IRX ITERVEC VR0;ERROR;SLOPE;ΔVR
[1]    VR←VR0
[2]    →L1×1E¯6<⌈/|ERROR←IRX-COM VR
[3]   L1:SLOPE←DIFCOM VR
[4]    VR←VR+ΔVR←ERROR÷SLOPE
[5]    →2
     ∇
```

For the sequence of **voltage** excitations described by a vector *V* the current response *I* of the component *COM* is directly calculable:

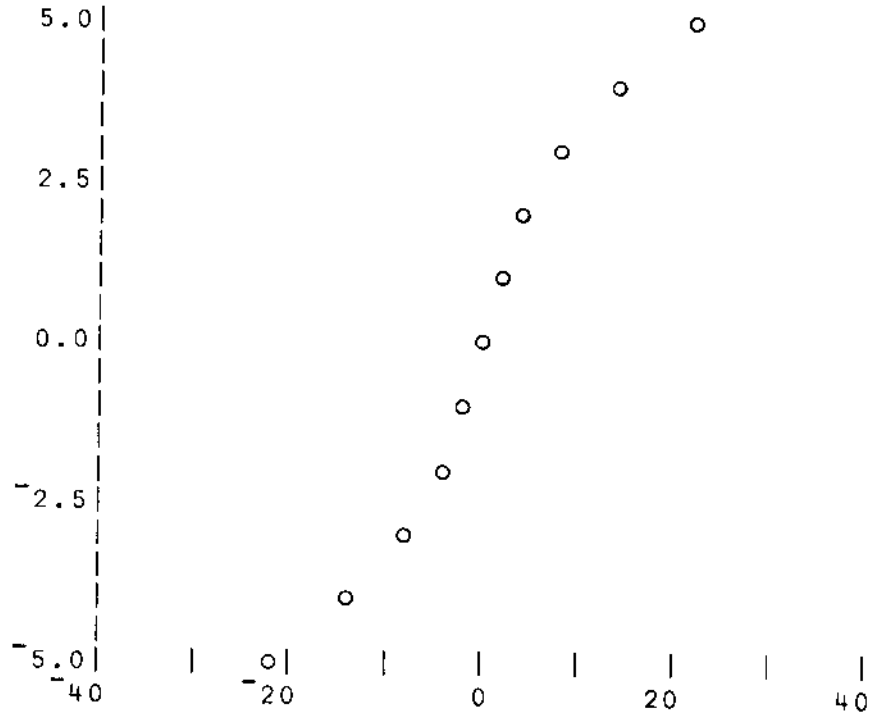
```
      □←V←¯6+⍳11
¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  3  4  5
      □←I←COM V
¯22.5 ¯14.4 ¯8.7 ¯4.8 ¯2.1  0  2.1  4.8  8.7  14.4  22.5
```

If this vector is now employed to describe a sequence of **current** excitations applied to the component, and an arbitrary guess is made at the corresponding vector of response voltages, the function *ITERVEC*

```
      I ITERVEC ⍳11?1000
¯5 ¯4 ¯3 ¯2 ¯1  0  1  2  3  4  5
```

returns a result which demonstrates that it has obtained, for the chosen excitation vector, the **inverse** of the function *COM*. Appreciation of this may be aided by a plot of *V* versus *I*, and comparison with the earlier plot of *I* versus *V*.

```
      30 40 PLOT (I ITERVEC ⍳11?1000) VS I
```



11.2 NONLINEAR CIRCUITS

The foregoing algorithm for the iterative analysis of a nonlinear component with a single excitation and one unknown voltage can easily be generalized to handle a nonlinear circuit containing more than one component, more than one current excitation, and more than one unknown voltage. More concisely, *IRX* and *VR* are now vectors rather than scalars.

The generalization is particularly clear if we recall one idea that has been repeatedly reinforced throughout this text: that there is no difference in the functional description of a circuit and a component. For this reason, the circuit to be used as an example in this section will first be considered as a 3-terminal component (Fig. 11.3), with no attention paid to its internal detail. Again, we assume current excitation to be applied at the terminals of the circuit (Fig. 11.3). The principal difference between the situations shown in Figs. 11.1 and 11.3 is that excitation is now applied to more than one terminal and that more than one terminal voltage is to be computed. Thus, *ERROR* is now a vector, as is the associated correction ΔVR to the vector *VR* of terminal voltages.

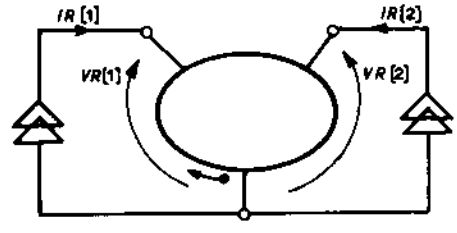


Fig. 11.3 A 3-terminal nonlinear circuit.

In the following section we consider the calculation of *ERROR*, which is the difference between the actual nodal injected currents and those which would be demanded by the assumed nodal voltages. In Section 11.4 we consider the task of computing any correction to *VR* that may be necessary. It is then possible, in Section 11.5, to describe and test an iterative algorithm for nonlinear circuit analysis.

11.3 THE BRANCH CONDUCTANCE ARRAY

The function *NET* relates the currents and voltages of a nonlinear circuit by the expression

```
IR←N NET VR
```

where *IR* and *VR* are the reduced vectors of nodal current and voltage respectively (Fig. 11.3), and *N* describes the circuit. We intentionally do not divulge the algorithm

associated with *NET*, so that we can concentrate upon essentials. For any guessed or corrected value of *VR* occurring in the iterative analysis of a nonlinear circuit, the function *NET* can calculate the corresponding value of injected nodal currents and, therefore, the difference (*ERROR*) between these and the **actual** injected nodal currents:

```
ERROR←IRX-(N NET VR)
```

Since *ERROR* is a vector, a test on the maximum reduction of the magnitude of its elements must determine whether the current value of *VR* can be taken as the actual vector of nodal voltages. In other words, corresponding to line 2 of *ITER*, we now have

```
→L1×1E¯6<⌈/|ERROR←IRX-(N NET VR)
```

The variable *N* which describes the circuit is obtained by means of a function *CIRCUIT*. The important property of this function is not its algorithm but the fact that it takes, as its arguments, the incidence matrix *A* and an array *T* which describes the individual components:

```
N←A CIRCUIT T
```

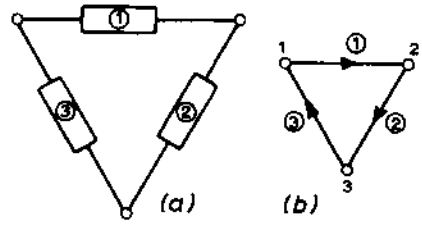
The function *CIRCUIT* is, in fact, no more than a convenient means of assembling, under a single name, the circuit properties separately described by *A* and *T*.

We are already familiar with the properties of the incidence matrix, and the fact that it is equally applicable to nonlinear as well as to linear circuits. The description of the nonlinear components by the array *T* is based on a power-series description of nonlinearity (first encountered in Chapter 2) and is best illustrated by example. Fig. 11.4a shows the interior detail of the circuit of Fig. 11.3, and the expressions describing the individual components. The collection of these components, numbered as shown, is described by the array *T*:

```
   T
2  0  4  1
0  0  0  0
0  0  0  0

0  0  0  0
3  1  2  0
0  0  0  0

0  0  0  0
0  0  0  0
1  6  0  0
```




```
IB[1]←(2×VB[1])+(4×VB[1]*3)+VB[1]*4
IB[2]←(3×VB[2])+(VB[2]*2)+2×VB[2]*3
IB[3]←VB[3]+6×VB[3]*2
```

Fig 11·4

Detail of the circuit of Fig. 11·3, and its directed graph.

The second plane of *T*, for example, shows that the corresponding branch current *IB*[2] is dependent upon the voltage *VB*[2] of branch 2 (row 2) raised to the powers from 1 to 4 (there are four columns) and multiplied by the coefficients associated with the various columns. The array *T* will be called the <u>branch conductance array</u>.

The above branch conductance array *T* is combined with the incidence matrix describing the directed graph of Fig. 11.4b

```
      □←A←INCID ⍉3 2 ⍴ 1 2 2 3 3 1
 1  0 ¯1
¯1  1  0
 0 ¯1  1
```

```
      N←A CIRCUIT T
```

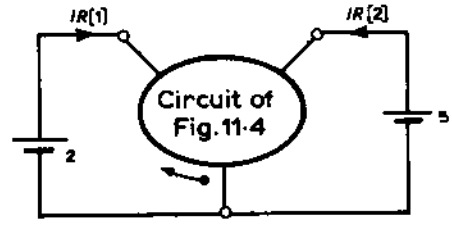


Fig. 11-5

Since the detail of *N* is unimportant to our discussion, it will not be examined. It is sufficient at present to understand that the current response to the voltage excitation shown in Fig. 11.5 can be computed by the function *NET*, for which *N* is one of the arguments

```
      □←IR←N NET 2 5
¯55 323
```

The reader may verify that these currents are compatible with the known nodal voltages and the descriptions of the components. In so doing he can discover the essential nature of the function *NET*.

11.4 THE REDUCED INCREMENTAL NODAL CONDUCTANCE MATRIX

For the single component circuit of Fig. 11.1 a <u>scalar</u> change ΔVR in the voltage *VR* is required to effect a <u>scalar</u> change $\Delta IR \leftarrow ERROR$ in the nodal current. If ΔVR and ΔIR are sufficiently small they are related by the <u>linear</u> expression

$$\Delta IR \leftarrow SLOPE \times \Delta VR$$

where *SLOPE* is then the incremental conductance (local conductance) of the component for the relevant value of *VR*. Despite the fact that ΔIR may not be small, the inverse function

$$\Delta VR \leftarrow \Delta IR \div SLOPE$$

is nevertheless employed to compute the change ΔVR from knowledge of $\Delta IR \leftarrow ERROR$ (on line 4 of *ITER*).

The situation is essentially the same for a multi-component circuit. Here a vector change ΔVR in nodal voltages is required to effect a vector change $\Delta IR \leftarrow ERROR$ in the nodal currents. If the elements of ΔVR and ΔIR are sufficiently small they are related by the linear expression

$$\Delta IR \leftarrow SLOPE +.\times \Delta VR \qquad (11.1)$$

where *SLOPE* is a reduced incremental nodal conductance matrix of dimension $(N-1)$ by $(N-1)$, where N is the number of nodes. Again, despite the fact that every element of ΔIR may not be small, we still employ the inverse function $\Delta VR \leftarrow \Delta IR \boxminus SLOPE$ to compute the (vector) correction ΔVR from knowledge of the vector $\Delta IR \leftarrow ERROR$.

The name 'reduced incremental nodal conductance matrix' is assigned to the matrix *SLOPE* because it relates small changes in the nodal currents and voltages of a multiterminal nonlinear component, just as an incremental conductance (also called a local or small-signal conductance) describes the relation between small changes in the current and voltage of a 2-terminal nonlinear component (see Sections 2.5 and 5.3).

For a nonlinear circuit described by the array *N* the matrix *SLOPE* is computed by means of the function *DIFNET* which takes, as its arguments, the array *N* and the current value of *VR*:

```
SLOPE←N DIFNET VR
```

This expression corresponds to line 3 of *ITER* (Details of the function *DIFNET* will not be displayed here, since they are not pertinent to our present discussion of nonlinear circuit analysis, though the interested reader is referred to Table 11.1). It now follows by reference to equation (11.1) that, corresponding to line 4 of *ITER*, the corrected value of *VR* is now computed according to the relation $VR \leftarrow VR + \Delta VR \leftarrow ERROR \boxminus SLOPE$

11.5 ITERATIVE ANALYSIS OF A NONLINEAR CIRCUIT

The discussion of the two preceding sections suggests the following function for the iterative analysis of a nonlinear circuit:

The relations between small changes in the currents and voltages of a nonlinear circuit are compared, below, with the relations between the currents and voltages of a circuit containing only linear homogeneous components.

	LINEAR HOMOGENEOUS CIRCUITS (ACTUAL CURRENTS AND VOLTAGES)	NONLINEAR CIRCUITS (SMALL CHANGES IN CURRENTS AND VOLTAGES)
Kirchhoff's Current Law	`I←A+.×IB` (a)	`ΔI←A+.×ΔIB` (d)
Kirchhoff's Voltage Law	`VB←(⍉A)+.×V` (b)	`ΔVB←(⍉A)+.×ΔV` (e)
Branch Relations	`IB←YB+.×VB` (c)	`ΔIB←YBI+.×ΔVB` (f)
Reduced Nodal Relations	(a,b,c) yield `IR←YR+.×VR` where `YR←⌹1 ⌹1↓A+.×YB+.×⍉A`	(d,e,f) yield `ΔIR←SLOPE+.×ΔVR` (cf. eqn 11.1) where `SLOPE←⌹1 ⌹1↓A+.×YBI+.×⍉A`

Equations (d) and (e) have aready been mentioned in Section 7.9. If only 2-terminal nonlinear components are involved, the matrix *YBI* in (f) is a diagonal matrix; each diagonal element is the incremental conductance of the associated 2-terminal component for the existing value of the component voltage.

Table 11.1

```
    ∇ VR←IRX ITERNET VR0;ERROR;SLOPE;ΔVR
[1]   VR←VR0
[2]   →L1×1E¯6<⌈/|ERROR←IRX-(N NET VR)
[3]  L1:SLOPE←N DIFNET VR
[4]   VR←VR+ΔVR←ERROR⌹SLOPE
[5]   →2
    ∇
```

The most striking property of this function is its similarity to the function *ITER* for the analysis of a nonlinear 2-terminal component. The only essential changes are the maximum reduction performed on line 2 to test for termination, and the replacement of scalar division by matrix inversion on line 4. The similarity demonstrates with remarkable clarity the value of working in terms of arrays, and the fact that there is little that is conceptually new in proceeding from a single nonlinear component to a multiterminal nonlinear circuit. The procedure embodied in the function *ITERNET* is again called the Newton-Raphson algorithm.

Let us use the function *ITERNET* to analyze the voltage response of the circuit of Fig. 11.4 to the current excitation shown in Fig. 11.6. The branch conductance array *T* and the incidence matrix *A* have already been generated, as has the array *N* describing the circuit.

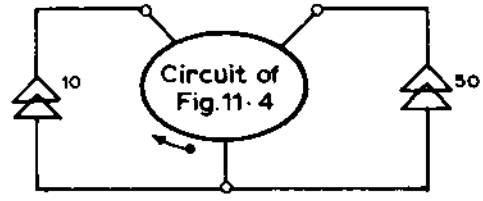


Fig. 11·6

We guess the reduced vector of nodal voltage response to be 1 6, and employ the function *ITERNET* to compute the actual value of this vector to within the tolerance determined by the test on line 2:

```
      □←VR←10 50 ITERNET 1 6
¯1.37 3.01
```

Thus, the voltages of nodes 1 and 2 with respect to the reference node (3) are ¯1.37 and 3.01 volts respectively.

The foregoing analysis will now be repeated with a trace on each line of *ITERNET*:

```
      T∆ITERNET←⍳5

      10 50 ITERNET 1 6

ITERNET[1] 1 6
ITERNET[2] 3
ITERNET[3]
    ¯209      198
     198       33
ITERNET[4] ¯0.37 4.12
ITERNET[5] 2
ITERNET[2] 3
ITERNET[3]
    ¯105      109
     109        3.58
ITERNET[4] ¯1.11 3.19
ITERNET[5] 2
ITERNET[2] 3
ITERNET[3]
     ¯80.2     ¯94.5
      94.5     ¯23.9
ITERNET[4] ¯1.36 3.01
ITERNET[5] 2
ITERNET[2] 3
ITERNET[3]
     ¯85.6     103
     103       ¯39.5
ITERNET[4] ¯1.37 3.01
ITERNET[5] 2
ITERNET[2] 3
ITERNET[3]
     ¯85.8     103
     103       ¯39.8
ITERNET[4] ¯1.37 3.01
ITERNET[5] 2
ITERNET[2] 0
¯1.37  3.01

      T∆ITERNET←⍳0
```

This computation shows that convergence to the final solution is quite rapid. It also draws attention to the fact that matrix inversion must be carried out at each iteration. This is significant in any consideration of the cost of nonlinear circuit analysis since, compared to other commonly used operations, matrix inversion is quite expensive.

For the simple nonlinear circuit of Fig. 11.1 the result of the iterative analysis was found to be independent of the initial guess, *VR0*, at the unknown voltage. At first, this appears to be the case for the circuit of Fig. 11.6:

```
      10 50 ITERNET 5 6
¯1.37  3.01
      10 50 ITERNET 20 ¯2
¯1.37  3.01
```

However, for different *VR0*, not far removed from those just employed,

```
      10 50 ITERNET 5 0
12.2  7.57
      10 50 ITERNET 12 7
12.2  7.57
```

the algorithm converges to a different solution which is, nevertheless, compatible with the nodal excitation:

```
      □←N NET 12.2 7.57
10  50
```

Thus, we have found by experiment that the nonlinear circuit of Fig. 11.6 does not possess a unique solution, but at least two solutions. As you may know, the fact that a nonlinear circuit can exist in two states is exploited in many practical circuits.

When discussing the behavior of linear circuits we found the superposition principle to be extremely valuable both in reducing the tedium of circuit analysis and in determining certain properties of linear circuits. Unfortunately, the principle does not apply to nonlinear circuits, as a simple example based on Fig. 11.6 will show:

```
      (10 0 ITERNET 1 6)+(0 50 ITERNET 1 6)
¯3.567839897  5.240400286
```

The resulting vector is not in agreement with the previously computed nodal voltage response.

The iterative procedure which we have used for the analysis of a nonlinear circuit, and which is based on the Newton-Raphson algorithm, is summarized in Fig. 11.7.

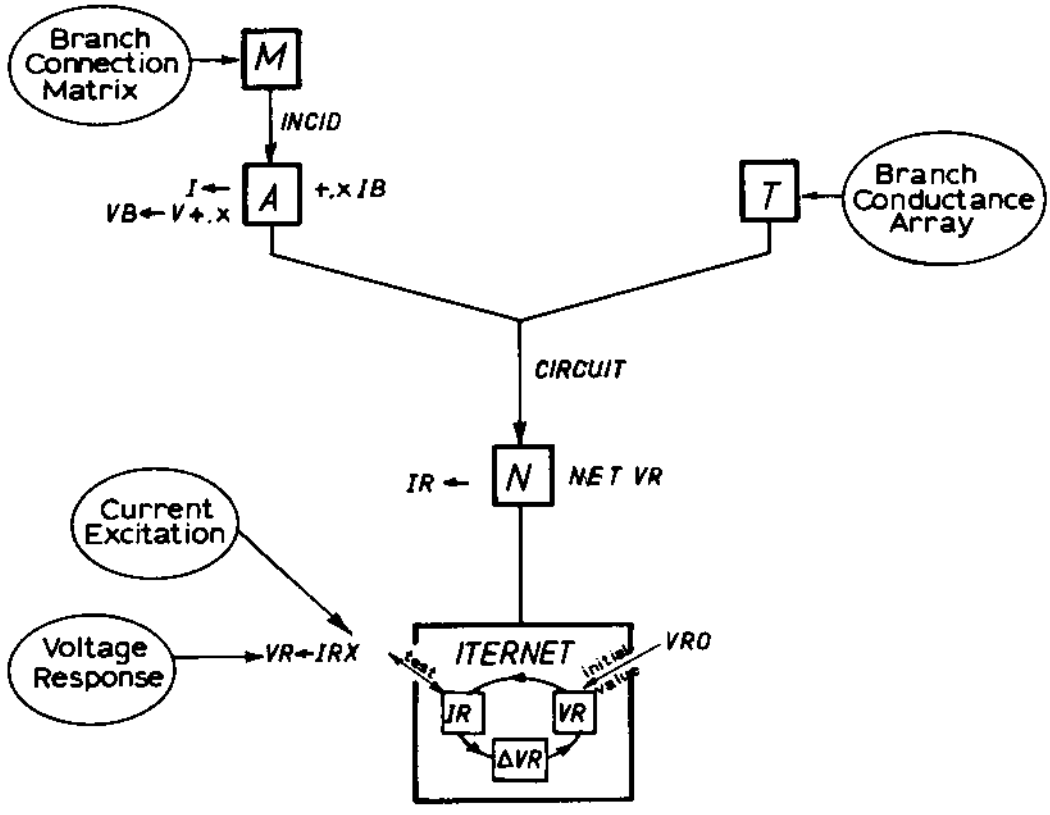


Fig. 11·7 Diagram showing the principal steps involved in nonlinear circuit analysis.

CHAPTER 12

SMALL-SIGNAL BEHAVIOUR OF NONLINEAR CIRCUITS

There is a widespread need in communication and instrumentation for some method of taking the output from a source whose signal component is small in amplitude and providing a much amplified version of that signal. The amplifier connected between a microphone and a loudspeaker (Fig. 12.1) is a typical example. In such a system what is important is the amplitude of the signal (i.e., time-varying) voltages and currents: of no interest to the system designer are the constant components of the circuit's voltages and currents.

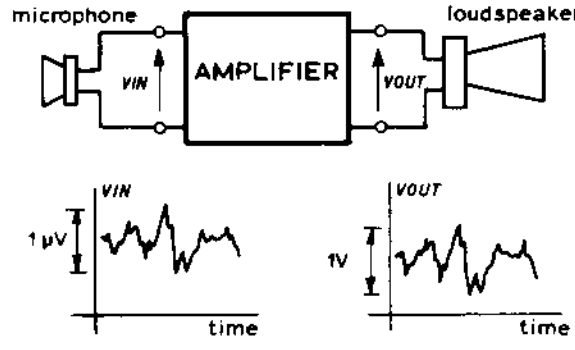


Fig. 12·1

With certain exceptions, the only practical way of achieving signal amplification is by the use of at least one 3-terminal component, of which the transistor is by far the most common. But the nonlinearity of the transistor prompts us to ask if its response to time-varying excitation must therefore be found by a series of nonlinear analyses such as were described in Chapter 11. If so, the determination of the response of a typical amplifier to signal excitation would be a time-consuming business. Fortunately, we need learn no new method of analysis; it will shortly be seen that the methods of component measurement and modelling and of circuit description and analysis discussed in Chapters 2 to 10 can be used, without modification, in the treatment of the small-signal response of nonlinear circuits.

12.1 SMALL-SIGNAL OPERATION

To see why this is so, we examine the voltage-current characteristics (Fig. 12.2) of a transistor which might, for example, be part of the amplifier of Fig. 12.1. When no signal is applied the amplifier's power supplies ensure that the constant values of the two transistor currents ($I[1]$ and $I[2]$) and voltages ($V[1]$ and $V[2]$) are represented by the two points labelled X in Fig. 12.2: the points X define the quiescent condition of the transistor. When a signal is applied to the amplifier, the point representing the condition of the transistor will move away from X,

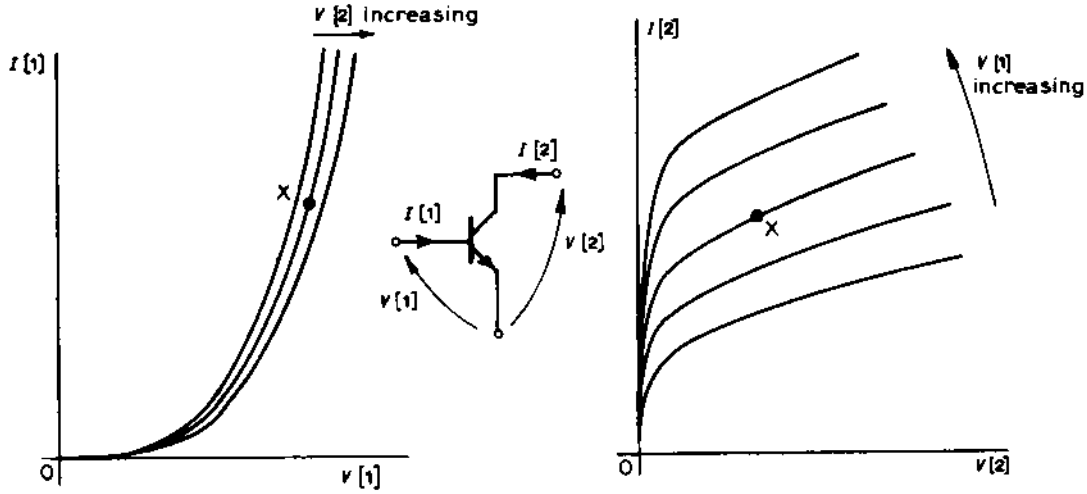


Fig. 12.2

The current-voltage characteristics of a transistor. The quiescent point is marked X. (*Note:* the spacing between the curves in the I[1], V[1] plane has been exaggerated).

consistently with the signal waveform, though it will return to X once the signal excitation is removed. We denote these departures from the constant values of $V[1]$, $I[1]$, $V[2]$, and $I[2]$ respectively by $\Delta V[1]$, $\Delta I[1]$, $\Delta V[2]$, and $\Delta I[2]$. However, if we are only interested in these departures from the quiescent condition caused by the signal, and not at all in the quiescent condition itself, it is helpful to insert appropriate axes, with their origin at X, on the voltage-current characteristics of Fig. 12.2: this is done in Fig. 12.3.

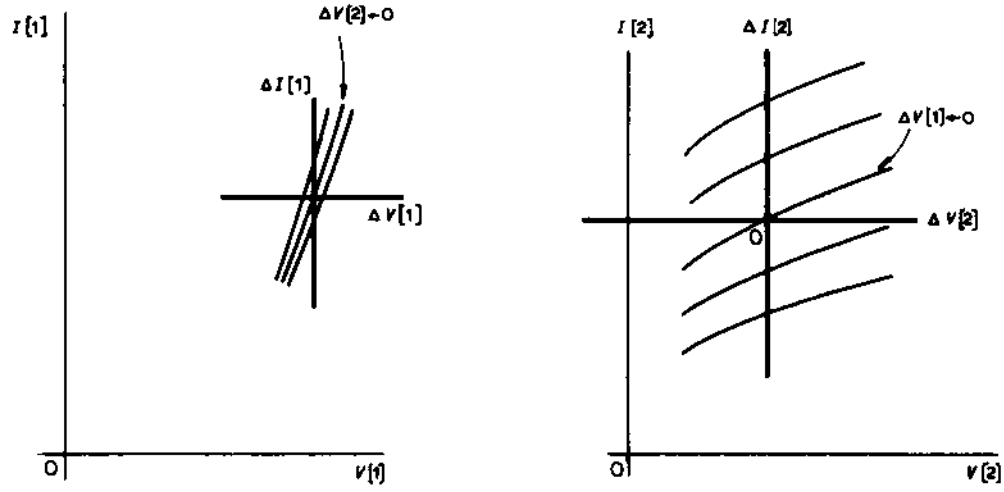


Fig. 12.3

Illustrating the approximate linear homogeneous relation between small deviations from the quiescent point X.

What immediately strikes us is the fact that in a region around X the relations between $\Delta V[1]$, $\Delta I[1]$, $\Delta V[2]$ and $\Delta I[2]$ are approximately **linear and homogeneous**. This is true even though the relations between $V[1]$, $I[1]$, $V[2]$ and $I[2]$ are **nonlinear**. In this case **all our previous experience is immediately relevant**: by recalling how the description of a collection of linear homogeneous components can be combined with a description of their interconnection to yield the description of the resulting circuit, we can now consider employing the same method to obtain a **small-signal** description of a nonlinear circuit. We are, of course, already familiar with the idea of the small-signal description of a nonlinear circuit (Section 11.4, equation 11.1), although we have not yet seen how it can be obtained from a knowledge of the components and their interconnection. That is the task of this chapter.

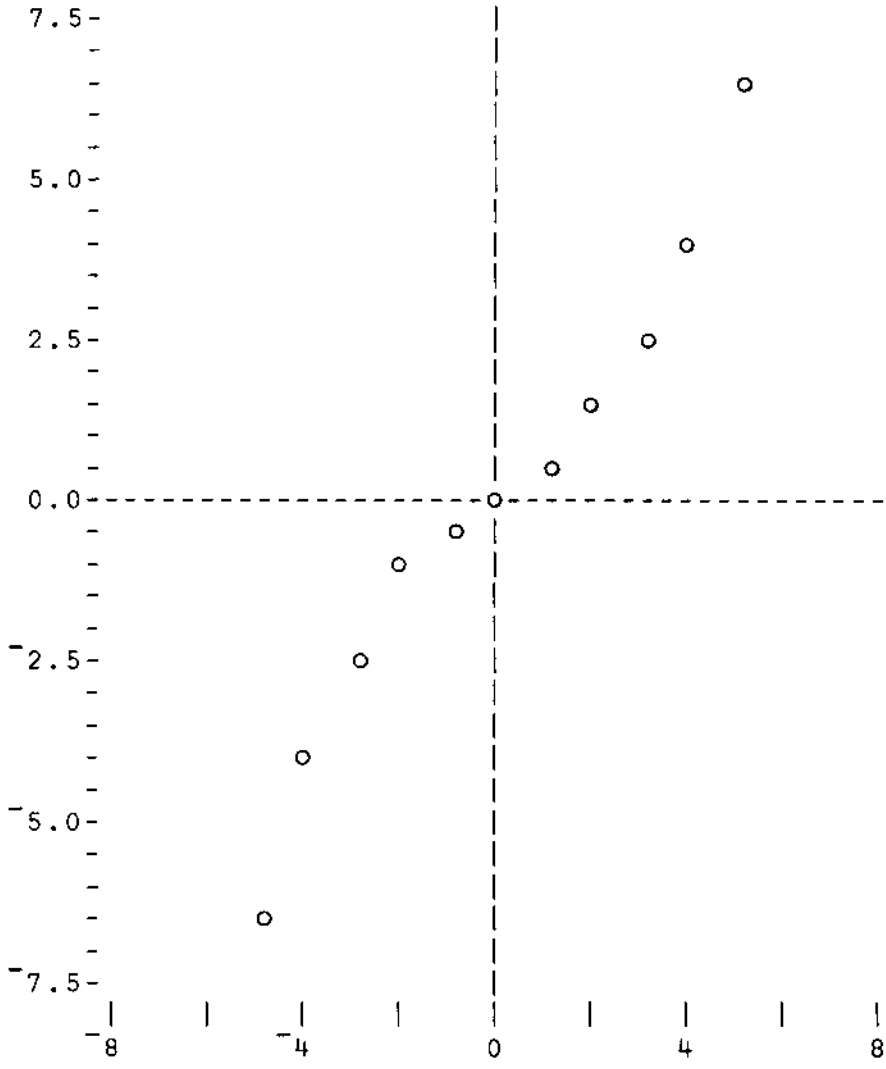
12.2 LOCAL CONDUCTANCE OF 2-TERMINAL COMPONENTS

In considering the relation between current changes and voltage changes in a 2-terminal component we shall, to some extent, be revising the material of Chapter 5. Consider the 2-terminal component described by the function *ONEV3*

```
    ∇ I←ONEV3 V
[1]   I←(0.5×V)+(÷32)×V*3
    ∇
```

whose current-voltage characteristic is shown below:

```
      V←¯6+⍳11
      I←ONEV3 V
      30 30 PLOT I VS V
```



In the absence of a signal, let the constant voltage excitation of *ONEV3* be 3 volts (Fig. 12.4), so that the current *I* is 2.344 amps:

```
      □←I←ONEV3 V←3
2.344
```

If the presence of a signal (Fig. 12.5) ensures that the voltage applied to the component varies by plus and minus one percent about the quiescent value, the extreme values of *V* are

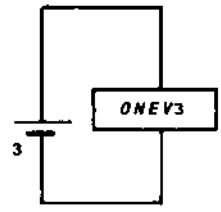


Fig.12·4

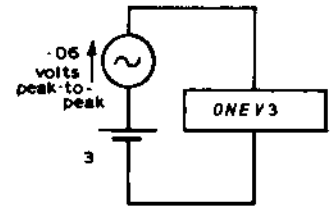


Fig. 12·5

```
      □←VE←V×(1.01 0.99)
3.03 2.97
```

If *V* changes then so does *I*, and the corresponding extreme values of *I* are

```
      □←IE←ONEV3 VE
2.384 2.304
```

The total excursions ΔV and ΔI in *V* and *I* respectively are therefore

```
      □←ΔV←-/VE
0.06
```

```
      □←ΔI←-/IE
0.08063
```

so that the slope of the function *ONEV3* (i.e., the local conductance) at *V*←3 is

```
      □←SLOPE←ΔI÷ΔV
1.344
```

Here we have shown how the slope of a function at a given point can be computed by considering very small changes in current and voltage, and without detailed knowledge of the function.

To check that the relation between changes in current and voltage is approximately linear and homogeneous over a region of the component's characteristic, we consider a sequence of excursions in voltage consisting of 1%, 2%, 3% and 4% of the quiescent value,

```
      A←0.01 0.02 0.03 0.04
```

so that the extreme values of V are $V\times 1+A$ and $V\times 1-A$. By forming the matrix

```
      (2,⍴A)⍴(1+A),1-A
1.01        1.02        1.03        1.04
0.99        0.98        0.97        0.96
```

we can generate an array of voltage extrema

```
      □←VE←V∘.×⍉(2,⍴A)⍴(1+A),1-A
3.03        2.97
3.06        2.94
3.09        2.91
3.12        2.88
```

for which the associated extrema of component current are

```
      □←IE←ONEV3 VE
2.384       2.304
2.425       2.264
2.467       2.225
2.509       2.186
```

Minus reduction over the last dimension now yields the vectors ΔV and ΔI from which the local conductance can be computed:

```
      □←ΔV←-/VE
0.06 0.12 0.18 0.25
      □←ΔI←-/IE
0.08063 0.1613 0.2419 0.3226
      □←GG←ΔI÷ΔV
1.344 1.344 1.344 1.344
```

Thus, it is evident that for sufficiently small excursions in voltage and current about the quiescent point, an approximately linear homogeneous relation exists between these excursions and is described by a local conductance (or **small-signal conductance**) of 1.344 mho. The signal voltage response to signal current excitation can be determined by means of the small-signal equivalent circuit shown in Fig. 12.6. Repetition of the above calculation for a range of quiescent conditions provides a reminder that for a nonlinear component the local conductance is a function of the quiescent condition:

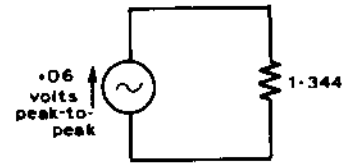


Fig. 12·6

```
      V←⍳10
      ΔV←-/VE←V∘.×(1.01 0.99)
      ΔI←-/IE←ONEV3 VE
      □←GG←ΔI÷ΔV
0.5938 0.875 1.344 2 2.844 3.875 5.094 6.5 8.094 9.875
```

Since it will often be necessary to determine the local conductance of a 2-terminal component at a specified quiescent condition, we define a function for this purpose. The algorithm is suggested by the foregoing discussion:

```
    ∇ G←Q LOCALG FN;VE;IE
[1]   VE←Q×(1.01 0.99)
[2]   IE←⍎FN,' VE'
[3]   G←(-/IE)÷(-/VE)
    ∇
```

A test of the function with the component and quiescent voltage (3 volts) considered in the text returns the expected value of local conductance:

```
      3 LOCALG 'ONEV3'
1.344
```

12.3 SMALL-SIGNAL DESCRIPTION OF SOURCES

We have shown that an approximately linear homogeneous relation

$$\Delta I \leftarrow GG \times \Delta V$$

exists between small changes in the voltage and current of a 2-terminal component, a relation that can be represented by a small-signal equivalent linear conductance. If we can discover similar small-signal equivalents for all the components of a circuit, we can use the method of circuit description and analysis described in chapters 8 and 9 to compute the relation between the signal currents and voltages associated with a circuit. We consider, first, the small-signal equivalents of ideal sources.

The source associated with a circuit will, in general, generate a voltage or current consisting of a constant component and a time-varying component, for which one of the representations of Fig. 12.7 is relevant. Just as a constant voltage or current was in earlier chapters regarded as excitation, the change ΔV or ΔI now constitutes the (signal) excitation. What about the constant sources appearing in Fig. 12.7? They are 2-terminal components, so we must find the relation they impose upon changes in current and voltage.

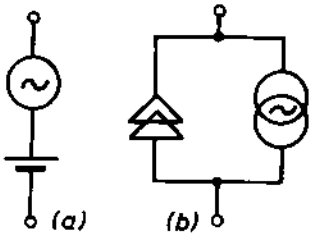


Fig. 12.7

For the constant voltage source appearing in Fig. 12.7a, and repeated in Fig. 12.8, it is clear that the change in voltage is, by definition, zero, that is, $\Delta V \leftarrow 0$ independently

of the change in current, ΔI. In other words, to signal voltages and currents, the constant voltage source is a short circuit. It is readily seen that a constant current source, for which $\Delta I \leftarrow 0$, may be represented, as far as signals are concerned, by an open circuit. This conclusion is confirmed by application of the function *LOCALG* to a function *ISOURCE* which describes a constant current source:

```
    ∇ I←ISOURCE V
[1]   I←10
    ∇

      (?100) LOCALG 'ISOURCE'
0
```

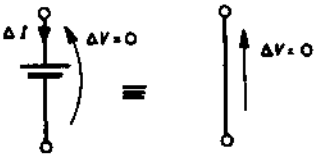


Fig. 12·8

12.4 AN EXAMPLE

The series connection of a linear resistor and an exponential diode is connected (Fig. 12.9) to a source whose open-circuit voltage comprises a constant component of 10 volts and a (zero-average) timevarying component of square waveform whose peak-to-peak amplitude is one volt. The internal resistance of this source is 1 ohm. We wish to compute the waveform of the voltage *VD* across the diode.

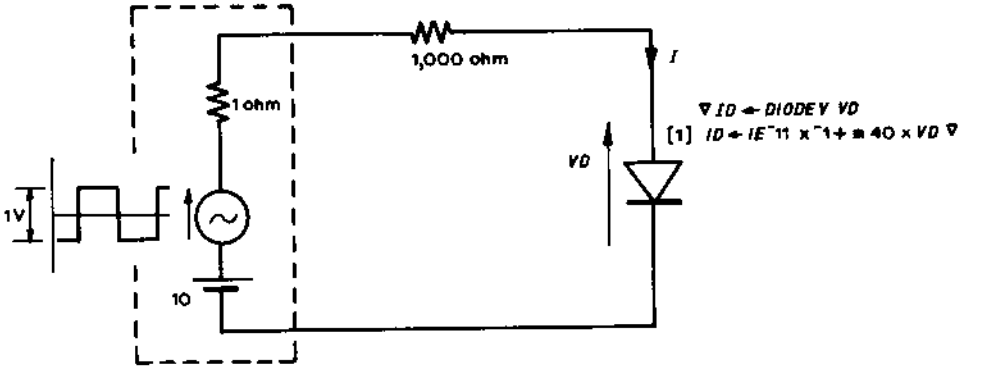


Fig. 12·9

To solve this problem we must eventually discover the relation between the signal current and voltage of each component. However, because these relations are in general dependent upon the circuit's quiescent condition, this must first be determined. To do so we set the signal excitation to zero (Fig. 12.10).

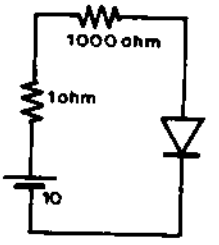


Fig. 12.10
Quiescent condition of the circuit of Fig. 12.9

The calculation of the circuit's quiescent condition can be performed in a number of ways: by a load-line construction; by guessing some values of *VD* and working back to see which corresponds to the known excitation of 10 volts; or, finally, by using an iterative scheme like that which is embodied in the function *ITER*. All these methods are illustrated in Fig. 12.11: the essential information they provide is the quiescent voltage of the diode, which is the only nonlinear component in the circuit: its value is 0.5167 volts.

Knowing the quiescent value of *VD*, we can now determine the small-signal description of the diode by means of the function *LOCALG*:

```
      □←GD←.5167 LOCALG 'DIODEV'
0.3827
```

Thus, if we are considering the small-signal behaviour of the diode at the known quiescent condition, it can be replaced by a linear homogeneous component whose conductance is 0.3827 mho. The linear components in Fig. 12.9 are also replaced by their small-signal equivalents which are, of course, conductances of the same value. The small-signal equivalent of the constant voltage source is a short circuit. Thus, for the known quiescent condition, the small-signal equivalent circuit of Fig. 12.12 describes the relations between the (small-amplitude) signal currents and voltages in the circuit of Fig. 12.9.

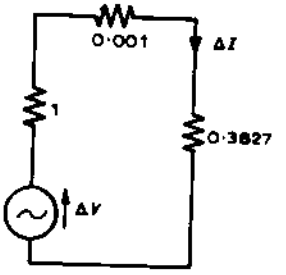


Fig. 12.12
Small-signal equivalent circuit of the circuit of Fig. 12.9

Load-Line Construction

Function *DIODEV* describes circuit to right of *AA'*, and function *SOURCE* describes circuit to left of *AA'*.

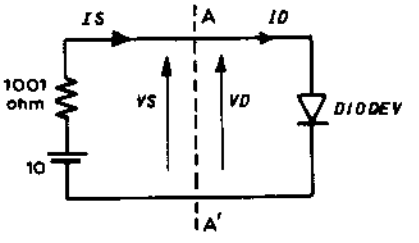
```
      ∇ ID←DIODEV VD
[1]   ID←1E¯11×¯1+*40×VD∇
```

```
      ∇ IS←SOURCE VS
[1]   IS←-(VS-10)÷1001 ∇
```

For a range of voltage adequate to include the actual value of *VD*, the load-line construction is easily obtained, and indicates a value of *VD* of approximately 0.52 volt.

```
      V←0,.05×⍳11
      IS←SOURCE V
      ID←DIODEV V

      30 60 PLOT IS AND ID VS V
```



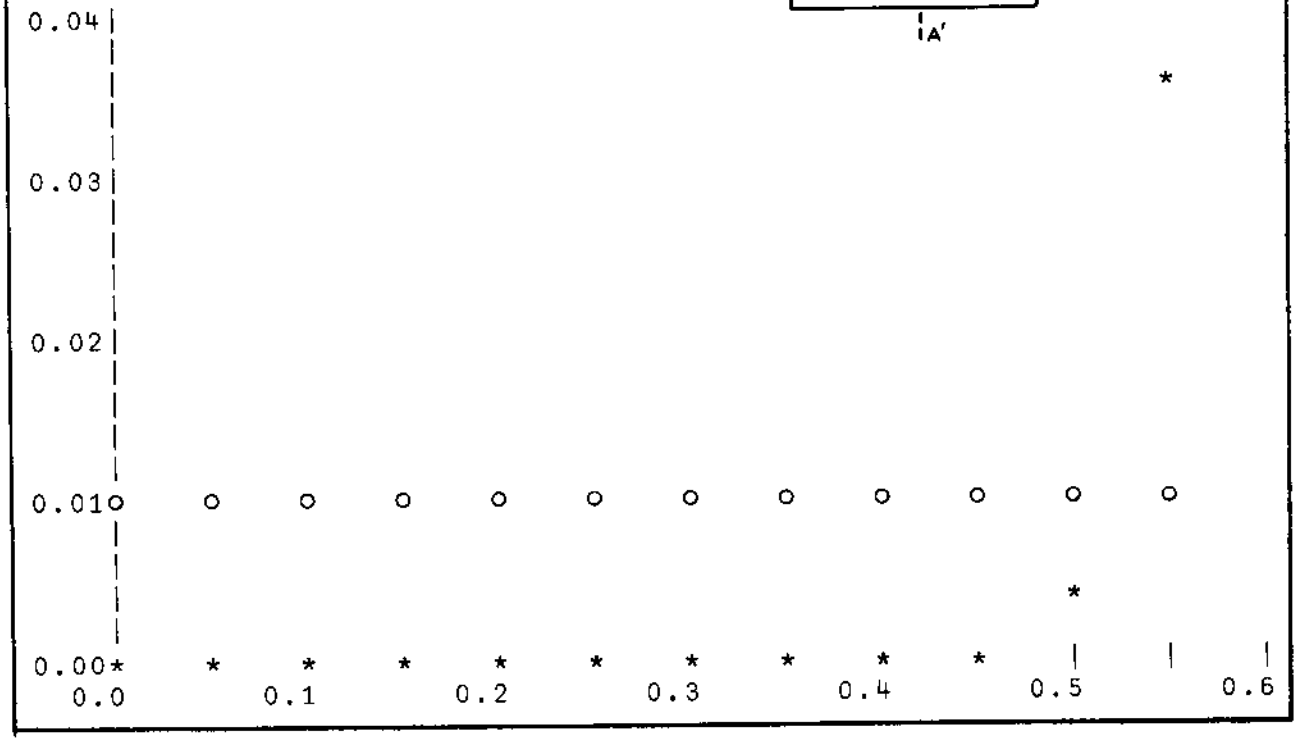


Fig. 12.11a

Analysis of the quiescent condition of the circuit of Fig. 12.10 by load-line construction

Exploration

Based on the result of the above calculation, we explore a narrow range of values of *VD*, and work back to the voltage across the diode and 1001 ohm resistance in series, and which we know is 10 volts. Examination of the plot shows that the actual value of *VD* is approximately 0.517 volt.

```
VD←.46+.01×⍳8
I←DIODEV VD
V←VD+I×1001

30 60 PLOT V VS VD
```

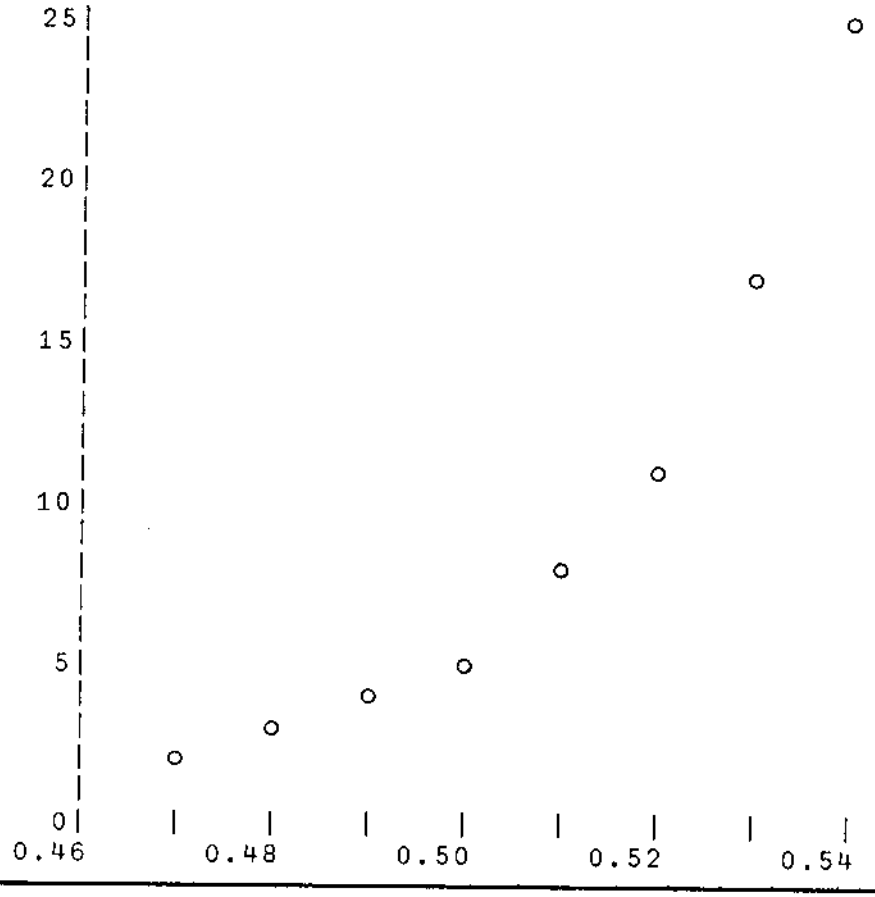
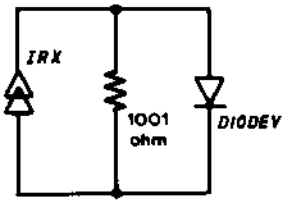


Fig. 12.11b

Analysis of the quiescent condition of the circuit of Fig. 12.10 by exploration

□□ **Iterative Analysis** (See Exercise E.11.14)
□□

By means of an iterative function (*EITER*) for the analysis of circuits containing both linear resistances and components described by exponential nonlinearity (and similar in nature to the function *ITER*) the Norton equivalent of the circuit (see figure) is analyzed:




```
      IRX←10÷1000
      □←VD←IRX EITER .5
0.5167
```

Fig. 12.11c

Analysis of the quiescent condition of the circuit of Fig. 12.10 by iterative analysis

This circuit is easy to analyze. The total resistance connected to the (signal voltage) excitation is

```
      R←1 RSS (÷0.001)RSS ÷GD
```

We know the peak-to-peak signal voltage excitation $\Delta V \leftarrow 1$ and can therefore determine the peak-to-peak value of the signal current response ΔI and the peak-to-peak voltage ΔVD across the diode:

```
      □←ΔI←ΔV÷R
0.0009964
      □←ΔVD←(÷GD)×ΔI
0.002603
```

Since the waveform of the excitation signal was square, the waveform of *VD* will also be square.* We can now combine our knowledge of the quiescent and signal behaviour, and plot the waveform of the diode voltage (Fig. 12.13).

---

* As in Chapter 5, when assuming a perfect square waveform for the circuit response we ignore energy storage effects (due to capacitance and inductance) which are always present.

The problem originally posed has been solved, with some precision, by a reasoned step-by-step method. But it is important to point out that it is this sort of simplification of a circuit which is done more or less automatically and intuitively by an experienced designer and which allows him to get a quick "feel" for the behaviour of a circuit. However, this sort of intuition is built up by repeated detailed analyses and by the development of a range of approximate equivalences for a variety of components and quiescent conditions.

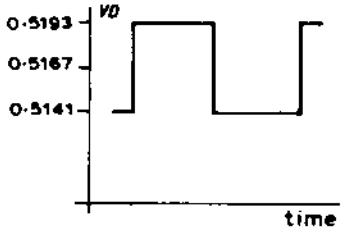


Fig 12·13

Waveform of the voltage of the diode in Fig. 12·9

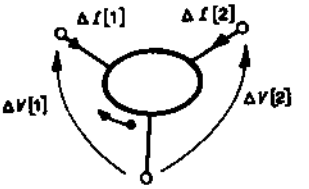


Fig. 12·14

12.5 3-TERMINAL COMPONENTS

In Chapter 6 we examined the relation between the total currents and voltages of a linear homogeneous 3-terminal component. We now seek, for a **nonlinear** 3-terminal component, an expression of the form

$$\Delta I \leftarrow GG +.\times \Delta V \qquad (12.1)$$

relating changes in the currents and voltages. GG is here a 2 by 2 **local conductance matrix** relating a 2-element vector ΔI of terminal current changes to a 2-element vector ΔV of terminal voltage changes. In parallel with Chapter 6, where we explicitly considered only two currents and two voltages, we now explicitly consider only the changes in these quantities (Fig. 12.14): the remaining current change and voltage change can be calculated from Kirchhoff's laws.

Consider the nonlinear 3-terminal component described by the function *TWOV6* which takes as its single argument an array of voltage excitation whose first dimension is 2. Arbitrarily, we choose the vector

```
      VQ←2 4
```

to define its quiescent condition. The corresponding vector *IQ* of quiescent terminal currents is

```
      □←IQ←TWOV6 VQ
24 10.8
```

We explore the current-voltage relation of the 3-terminal component in exactly the same way as in chapter 6, but now choosing the ranges of *V*[1] and *V*[2] to have *VQ*[1] and *VQ*[2], respectively, as their mid-point:

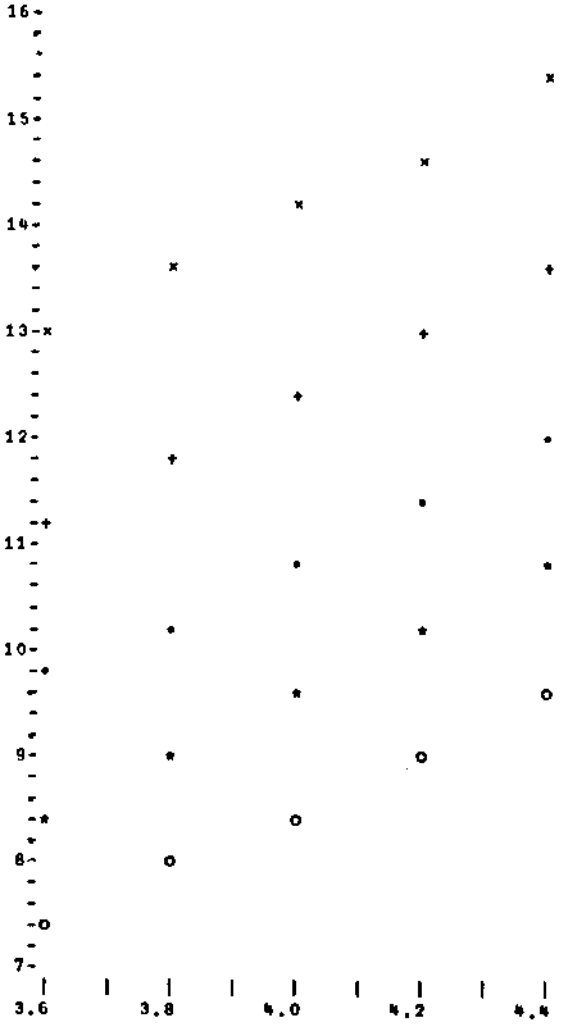
```
      □←V1←VQ[1]+0.2×¯3+⍳5
1.6 1.8 2 2.2 2.4
      □←V2←VQ[2]+0.2×¯3+⍳5
3.6 3.8 4 4.2 4.4
```

A 3-dimensional excitation array *V* gives rise to a current response *I* as follows:

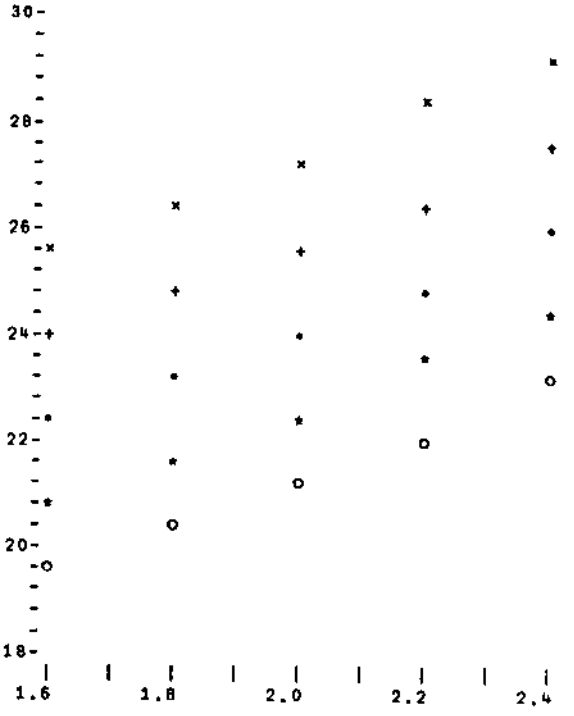
```
      V←V1 PP V2
      I←TWOV6 V
```

The same type of plot examined in Chapter 6 now enables some nonlinearity to be detected, though a later, more detailed examination will show the nonlinearity more clearly.

40 40 *PLOT* (⍟*I*[2;;]) *VS* *V*2

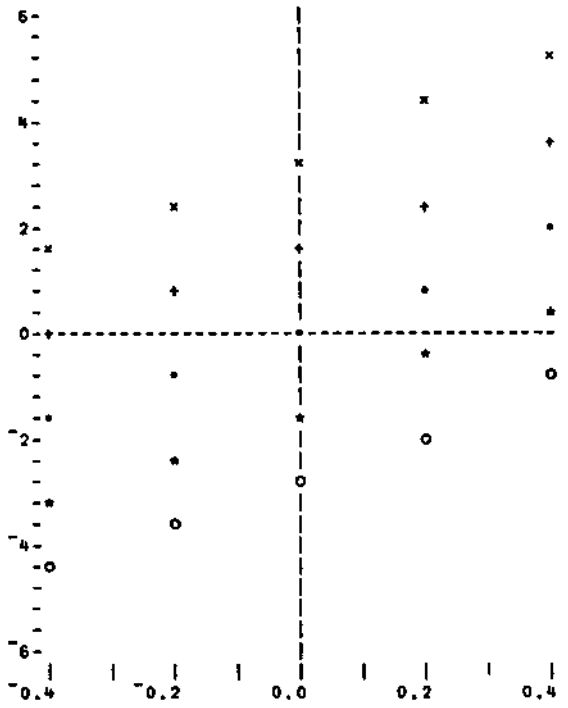


40 40 *PLOT* *I*[1;;] *VS* *V*1

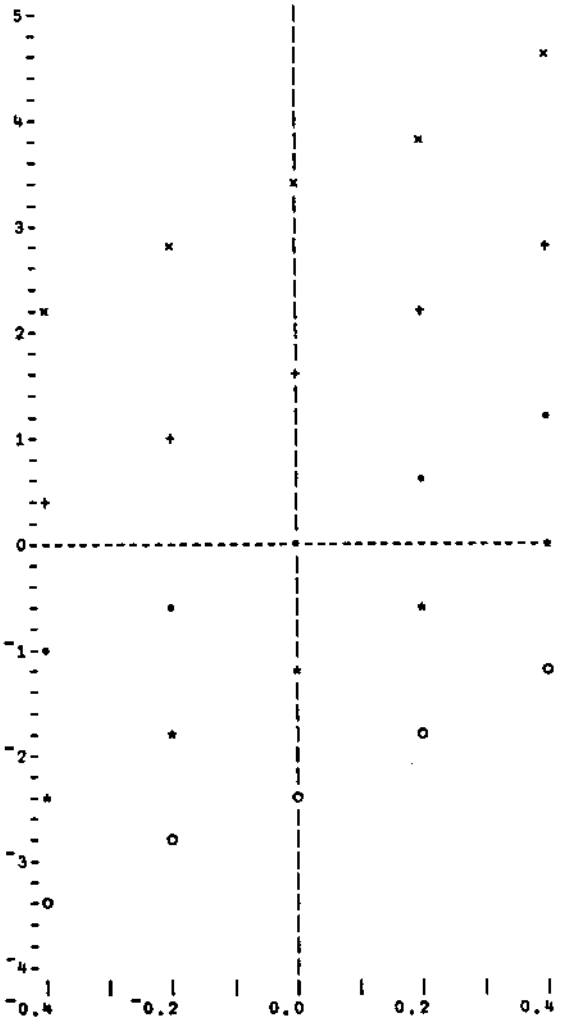


More conveniently, however, we can subtract the quiescent values from $I[1]$, $I[2]$, $V[1]$ and $V[2]$, for then the plots emphasize the approximately linear and homogeneous relation that exists between the changes ΔI and ΔV in current and voltage around the quiescent condition:

```
      40 40 PLOT (I[1;;]-IQ[1]) VS (V1-VQ[1])
```




```
      40 40 PLOT (⍉I[2;;]-IQ[2]) VS (V2-VQ[2])
```



This relation is approximated by the expression

```
ΔI←GG+.×ΔV
```

From Chapter 6 we know how to obtain the elements of the local conductance matrix *GG*. Use of the function *DIFF* places in evidence the effect upon *I*[1] of equal changes in *V*[1], while *V*[2] is held at its quiescent value (the third element of *V*2):

```
      DIFF I[1;;3]
0.747 0.834 0.93 1.04
```

This effect is clearly nonlinear, but can be approximated by a linear function if departure from the quiescent point is limited. The calculation

```
      (DIFF I[1;;3])÷DIFF V1
3.74 4.17 4.65 5.18
```

generates conductance parameters which relate changes in *I* and *V*: the parameter corresponding to the quiescent condition is the average of the second and third elements of this vector. Thus *GG*[1;1] equals 4.41. With reference to the plot of *I*[1] versus *V*[1], repeated with interpolation in Fig. 12.15a, *G*[1;1] is the slope at the quiescent point since (see equation 12.1) *G*[1;1] is the ratio Δ*I*[1]÷Δ*V*[1] evaluated for Δ*V*[2]←0, that is, with *V*[2] constant at its quiescent value.

Similarly, the average of the second and third elements of the vector

```
      (DIFF I[1;3;])÷DIFF V2
7.11 7.56 8.04 8.55
```

yields the value of the parameter *GG*[1;2], which is a measure of the effect, on *I*[1], of equal changes in *V*[2], with *V*[1] constant at its quiescent value. Thus *GG*[1;2] equals 7.8. From Fig. 12.15b it is seen that this parameter can be calculated from the spacing of the curves, since (see equation 12.1) *GG*[1;2] is the ratio Δ*I*[1]÷Δ*V*[2] evaluated with *V*[1] constant at its quiescent value. In a similar manner the remaining elements of the matrix *GG* can be found, and the matrix defined:

```
      (DIFF I[2;;3])÷DIFF V1
5.34 6.42 7.62 8.94
      (DIFF I[2;3;])÷DIFF V2
2.45 2.68 2.92 3.17
```

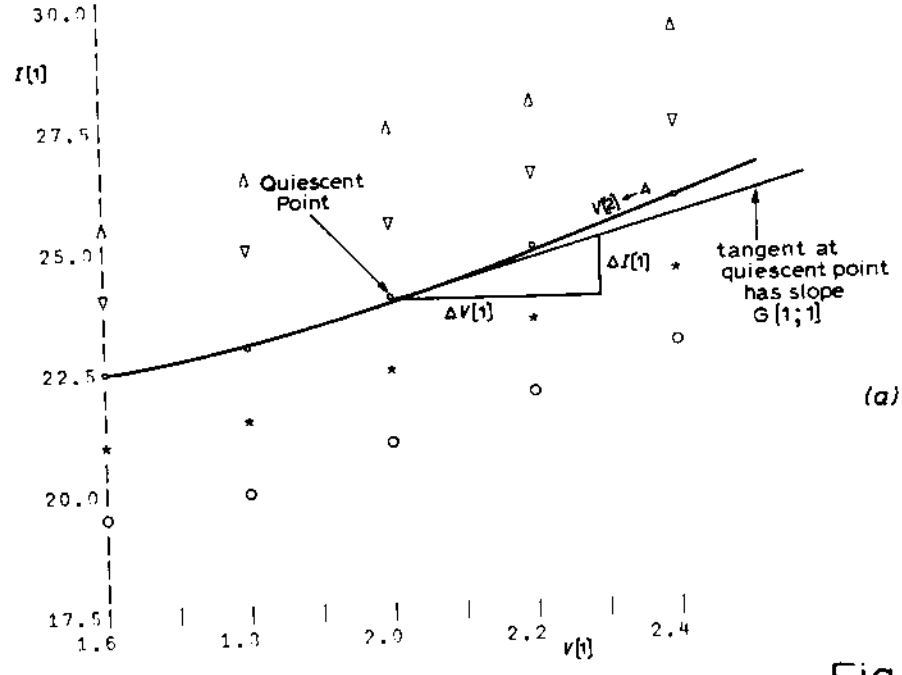
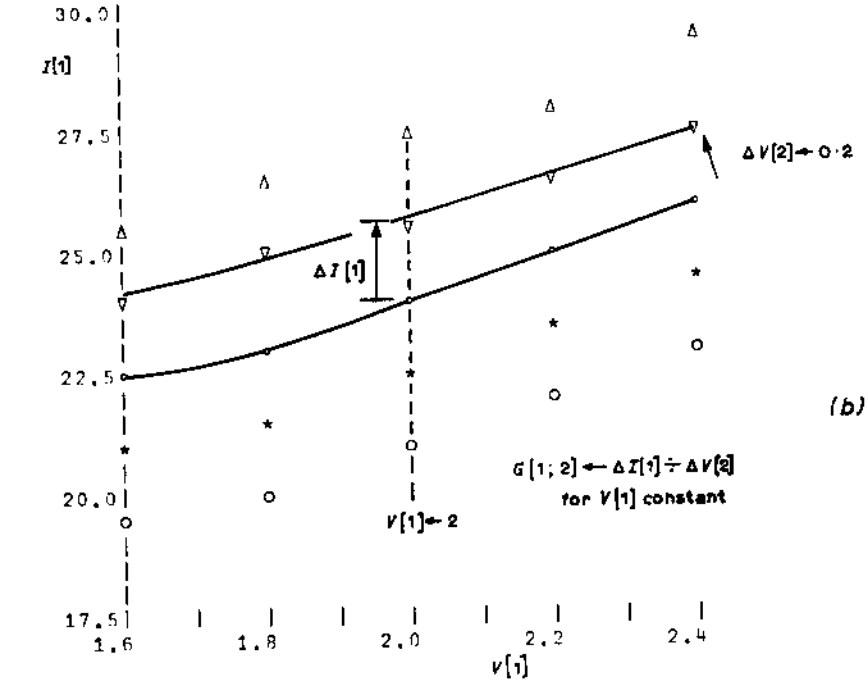


Fig. 1215



Thus $GG[2;1]$ equals 7.02 and $GG[2;2]$ equals 2.8. From these four elements the matrix GG can be defined as follows:

```
      □←GG←2 2⍴4.41 7.8 7.02 2.8
4.41          7.8
7.02          2.8
```

Thus we have constructed a model

$$\Delta I \leftarrow GG +.\times \Delta V \qquad (12.1)$$

```
      ∇ I←MODEL V;VQ;IQ;GG;ΔV;ΔI
[1]     VQ←2 4
[2]     IQ←24 10.8
[3]     GG←2 2⍴4.41 7.80 7.02 2.80
[4]     ΔV←(V[1;;]-VQ[1]),[0.5](V[2;;]-VQ[2])
[5]     ΔI←GG+.×ΔV
[6]     I←(ΔI[1;;]+IQ[1]),[0.5](ΔI[2;;]+IQ[2])
      ∇
      V1←2+.1×¯11+⍳21
      V2←4+.1×¯11+⍳21
      V←V1 PP V2

' *'[1+.1<|(MODEL V)-TWOV6 V]   ' *'[1+.5<|(MODEL V)-TWOV6 V]
```

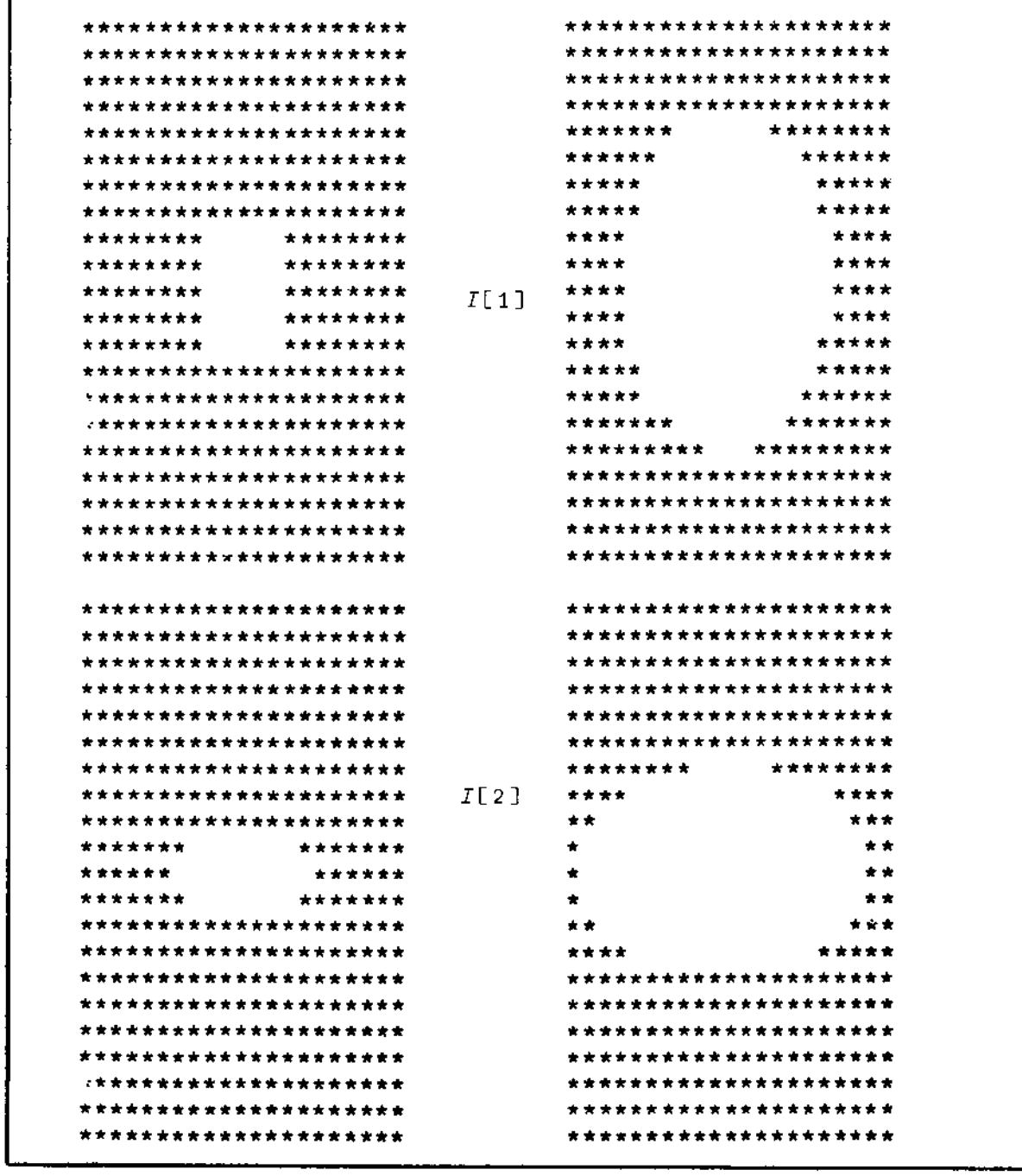


Figure 12.16

relating changes in I and V from their quiescent values IQ and VQ respectively, and which we expect to be valid to a reasonable accuracy in a region centred upon the quiescent condition. A test of the validity of this model is illustrated in Fig. 12.16.

Just as for the linear homogeneous components in Chapter 6, the vectors ΔI and ΔV in equation (12.1) are associated with two branches, as shown in Fig. 12.17. In other words, the same branch-node representation is appropriate to the total currents and voltages of a linear homogeneous 3-terminal component as well as to small changes in the currents and voltages of a nonlinear component.

In Fig. 12.17 and in equation (12.1) only the branch current changes and branch voltage changes associated with a single 3-terminal component were considered. If ΔIB and ΔVB are vectors whose elements account for all the branch current and voltage changes in a collection of components that together form a circuit, then a relation

$$\Delta IB \leftarrow YBL +.\times \Delta VB \qquad (12.2)$$

can be found which is similar in nature to the branch conductance matrix of a linear homogeneous circuit. The matrix YBL will be called the local branch conductance matrix.

Some components in an otherwise nonlinear circuit may, of course, be linear and homogeneous. In this case their conductance matrix is also their local conductance matrix.

In the analysis of small-signal circuit behaviour the calculation of the local conductance matrix of a 3-terminal component is a means to an end. We therefore define a function by means of which the local conductance matrix can be generated for any 3-terminal component and any quiescent condition:

```
    ∇ GG←Q LOCALGG FN
[1]   VQ←Q∘.×2 2⍴1
[2]   VE←VQ+(UNIT 2)∘.×.1 ¯.1
[3]   IE←⍎FN,' VE'
[4]   GG←(-/IE)+.×⌹-/VE
    ∇
```

It is left as an exercise for the student to validate this function - reference to equation (6.9) of Chapter 6 may help.

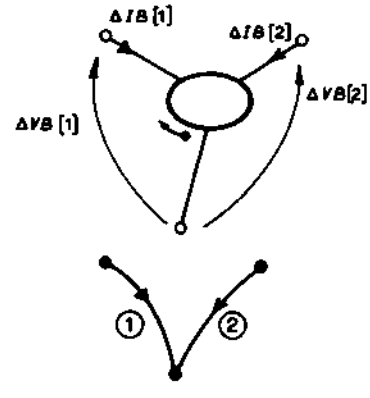


Fig. 12.17

12.6 SMALL-SIGNAL CIRCUIT ANALYSIS

In equation (12.2) we have the relation between small changes in the branch currents and voltages of a collection of possibly nonlinear components. We may also recall, from Section 7.9 of Chapter 7, that changes in current and voltage also obey Kirchhoff's laws:

```
ΔI←A+.×ΔIB
ΔVB←ΔV+.×A
```

where A is the familiar incidence matrix and ΔI and ΔV are now vectors describing changes in nodal currents and voltages. These three relations are shown in Table 11.1 together with the three relations governing the behaviour of circuits containing only linear homogeneous components.

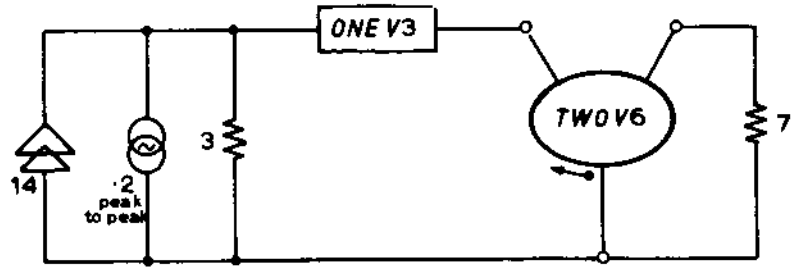


Fig. 12.18 Nonlinear circuit whose small-signal behaviour is to be found.

It is clear from this table that precisely the same methods that were discussed in Chapters 8 and 9 can now be used to calculate the small-signal behaviour of a nonlinear circuit: that is, the relation between ΔI and ΔV imposed by the circuit. Without more ado, then, we shall illustrate the small-signal analysis of a nonlinear circuit by means of an example.

The nonlinear circuit to be analysed is shown in Fig. 12.18 and contains, in addition to some linear components, the nonlinear components *ONEV3* and *TWOV6* encountered earlier. The circuit is excited both by a source of constant current and a source of signal current of square waveform; its peak-to-peak amplitude is 0.2 amps. The task is to compute the waveform of the signal voltage generated across the 7 mho conductance.

Solution of this problem involves three distinct and well defined steps, each of which has previously been discussed separately. They are, determination of the quiescent condition, local description of components, and small-signal circuit descriptions.

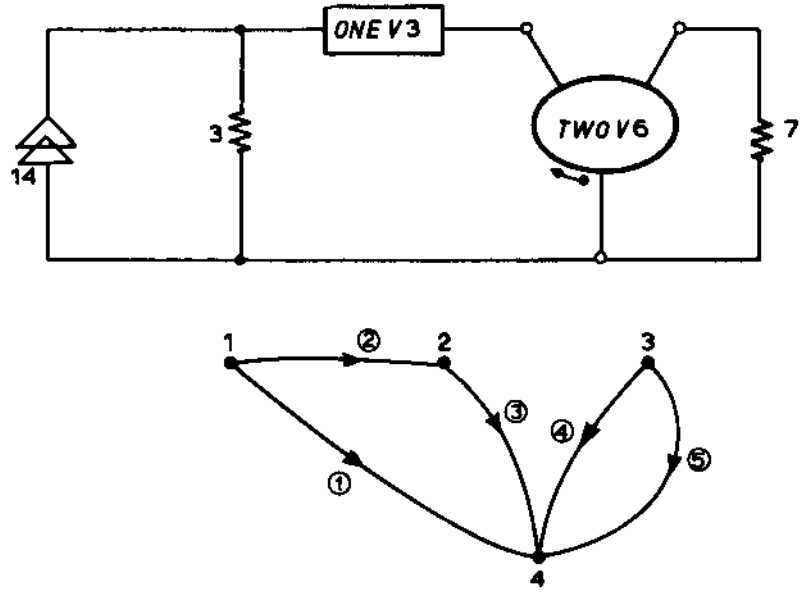


Fig. 12·19 Circuit for determination of the quiescent condition, and the directed graph pertinent to both the quiescent and small-signal analyses.

Determination of quiescent condition. The circuit (Fig. 12.19) for this calculation differs from the actual circuit only in that the signal excitation is set to zero. In order to carry out the quiescent analysis the directed graph of the circuit (Fig. 12.19) must be drawn and the branches and nodes numbered. Use of the function *ITERNET* developed in Chapter 11 is illustrated in Fig. 12.20, and yields the following vector of quiescent branch voltages:

```
      VB
3.998 2.735 1.263 ¯0.3067 ¯0.3067
```

Local description of components. We need to know the quiescent conditions of the nonlinear components in order to determine their description by a local conductance or local conductance matrix. There are two nonlinear components. For the function *ONEV3*, knowledge of its quiescent condition

```
      VB[2]
2.735
```

allows the function *LOCALG* to be used to determine its local conductance:

```
      □←X←VB[2] LOCALG 'ONEV3'
1.201
```

The functions *ONEV3* and *TWOV6* are:

```
     ∇ I←ONEV3 V
[1]    I←(0.5×V)+(÷32)×V*3
     ∇
```

```
     ∇ I←TWOV6 V
[1]    M←2 2⍴2 3 1 0.4
[2]    N←2 2⍴0.2 0.1 0.5 0.05
[3]    I←(M+.×V)+N+.×V*3
     ∇
```

The branch conductance array, incidence matrix and current excitation vector pertinent to the circuit of Fig. 12.19 are

```
      T                                    A
3     0     0                  1  ¯1   0   0   0
0     0     0                  0  ¯1   1   0   0
0     0     0                  0   0  ¯1   1   1
0     0     0                 ¯1   0  ¯1  ¯1  ¯1
0     0     0

                               IRX←14 0 0
0     0     0
0.5   0     0.0312
0     0     0
0     0     0
0     0     0

0     0     0
0     0     0
2     0     0.2
3     0     0.1
0     0     0

0     0     0
0     0     0
1     0     0.5
0.4   0     0.05
0     0     0

0     0     0
0     0     0
0     0     0
0     0     0
7     0     0
```

With respect to node 4, the quiescent nodal voltages are found by use of the function *ITERNET*:

```
      N←A CIRCUIT T
      □←VR←IRX ITERNET 0 0 0
3.998 1.263 ¯0.3067
```

The vector of quiescent branch voltages is found by invoking Kirchhoff's voltage law:

```
      □←VB←(⍉A)+.×VR,0
3.998 2.735 1.263 ¯0.3067 ¯0.3067
```

Fig. 12.20
Quiescent analysis of the circuit of Fig. 12.18

Thus, in the small-signal equivalent circuit, the component *ONEV*3 must be replaced by a linear homogeneous conductance of 1.201 mho. Similarly, the function *LOCALGG* is employed to generate the local conductance matrix of the component *TWOV*6 at its quiescent condition

```
      □←W←VB[3 4] LOCALGG 'TWOV6'
2.959          3.029
3.398          0.4146
```

Small-signal circuit description. Knowledge of the small-signal equivalent of each component allows the small-signal equivalent of the circuit to be drawn (Fig. 12.21): there is no need to draw its directed graph, for this is identical with that shown in Fig. 12.19. Note that the source of constant current has been replaced by an open circuit.

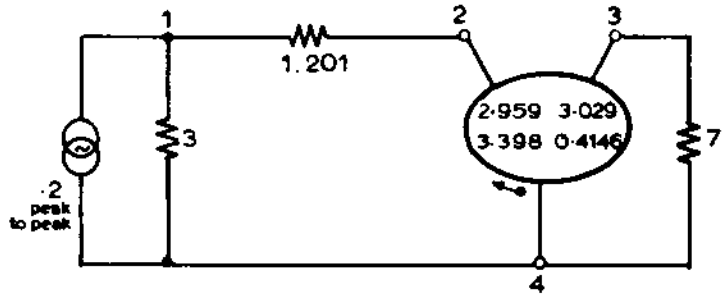


Fig. 12·21 Small-signal equivalent of the circuit of Fig. 12·18

The incidence matrix of the circuit of Fig. 12.21 is the same as for the original circuit:

```
      A
 1  ¯1   0   0   0
 0  ¯1   1   0   0
 0   0   0   1   1
¯1   0  ¯1  ¯1  ¯1
```

The local branch conductance matrix is formed from the branch relations of the components of the small-signal equivalent circuit:

```
      □←YBL←3 AA X AA W AA 7
3          0          0          0          0
0          1.201      0          0          0
0          0          2.959      3.029      0
0          0          3.398      0.4146     0
0          0          0          0          7
```

Just as for a circuit composed entirely of linear homogeneous components, the incidence matrix and local branch conductance matrix may be combined to yield a complete local nodal conductance matrix *YL* relating small changes in nodal currents and voltages:

```
ΔI←YL+.×ΔV
```

However, if the nodal small-signal current excitation is known, what is required is the reduced local nodal resistance matrix *ZRL* relating reduced vectors of small changes in nodal currents and voltages:

```
ΔVR←ZRL+.×ΔIR
```

In view of the similarity between the columns of Table 11.1, page 159, the previously defined function *ZEDR* may be used to calculate the matrix *ZRL*, under the assumption that the highest numbered node is the reference node:

```
ZRL←A ZEDR YBL
```

The signal excitation of the circuit of Fig. 12.18 (shown also in Fig. 12.21), is

```
ΔIR←.2 0 0
```

and therefore the reduced vector of nodal signal voltages (i.e., the signal voltage response) is

```
      □←ΔVR←ZRL+.×ΔIR
0.05434 0.2354 ¯0.01079
```

In combination with the reduced vector of quiescent nodal voltages (determined above - see Fig. 12.20):

```
      VR
3.998 1.263 ¯0.3067
```

we now have a record of the behaviour of each nodal voltage with respect to the reference node. Thus, the waveform of the voltage across the 7 ¯mho conductance (branch 5) has a mean value (*VR*[3]) of ¯0.3067 volts and a peak-to-peak amplitude (Δ*VR*[3]) of ¯0.01079 volts, and is shown in Fig. 12.22. The negative sign associated with the signal voltage means that this voltage decreases as the signal excitation increases (i.e., Δ*VB*[3] is negative when Δ*IR*[1] is positive), so that the response waveform is an inverted replica of the excitation waveform.

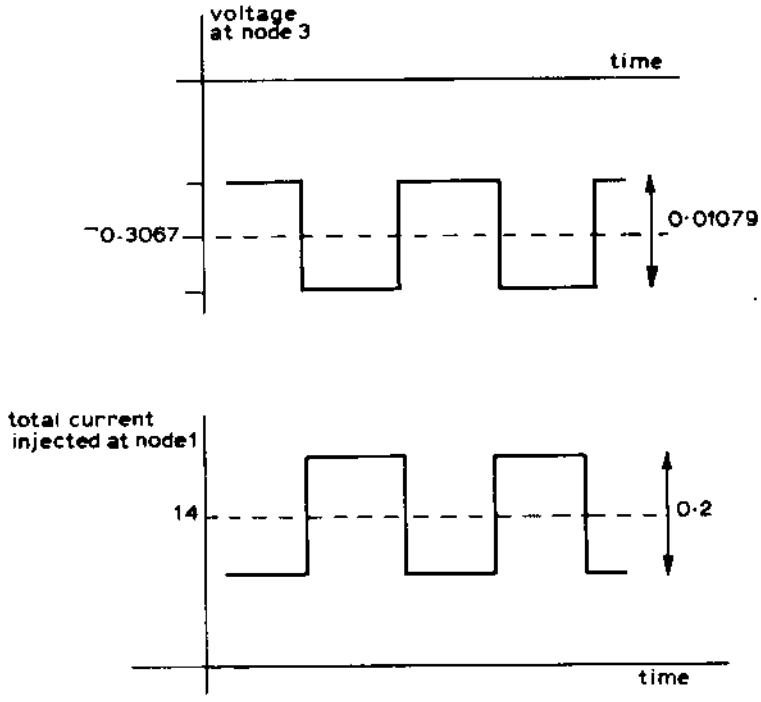


Fig. 12·22

Fig. 12.23 summarizes, in diagrammatic form, what is involved in the small-signal analysis of nonlinear circuits.

The discussion of the present chapter has been concerned with showing the practical value of the material of Chapters 8 and 9 in the analysis of the small-signal behaviour of circuits. Linear homogeneous components* having three or more terminals are rare, and the principal justification for their study is that they do describe, to a good approximation, the relations between signal voltages and currents of small amplitude.

---

* We exclude circuits composed of linear 2-terminal components, and which may be regarded as a 3-terminal component if there are three accessible terminals.

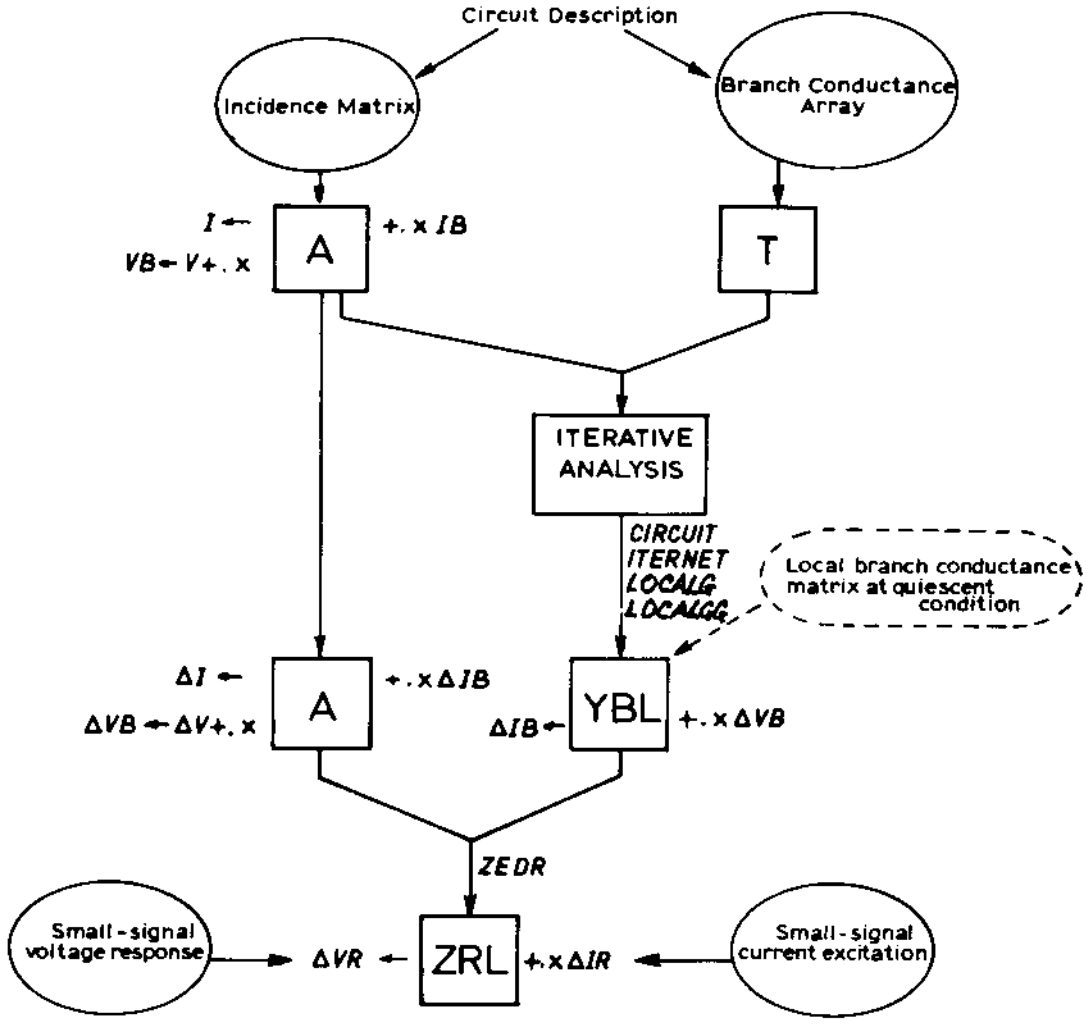


Fig. 12·23
Diagram showing the steps involved in the small-signal analysis of a nonlinear circuit.

12.7 CALCULATION OF SMALL-SIGNAL EQUIVALENTS IN THE COURSE OF NONLINEAR CIRCUIT ANALYSIS

The reader may have observed that the iterative procedure embodied in the function *ITERNET* depends, for its operation, upon the generation of a matrix *SLOPE* (see the function *DIFNET*) which is, in fact, the local reduced nodal conductance matrix of the circuit:

```
    ∇ VR←IRX ITERNET VR0;ERROR;SLOPE;ΔVR
[1]   VR←VR0
[2]   →L1×1E¯6<⌈/|ERROR←IRX-(N NET VR)
[3]   L1: SLOPE←N DIFNET VR
[4]   VR←VR+ΔVR←ERROR⌹SLOPE
[5]   →2
    ∇
```

Thus, in the course of analysing the quiescent behaviour of a circuit, the function *ITERNET* automatically generates one of the matrices from which the small-signal behaviour can be computed. This can be illustrated with reference to the circuit of Fig. 11.6, for which the branch array, incidence matrix and constant nodal current excitation are:

```
    T                    A          IRX←10 50
 2 0 4 1            ¯1  1  0
 0 0 0 0            ¯1  0  1
 0 0 0 0             0 ¯1 ¯1

 0 0 0 0
 3 1 2 0
 0 0 0 0

 0 0 0 0
 0 0 0 0
 1 6 0 0
```

Analysis to find the quiescent condition was easy to carry out:

```
N←A CIRCUIT T
VR←IRX ITERNET 1 6
```

Recalling that the function *DIFNET* takes the reduced vector of nodal voltages as its right argument and returns the reduced local nodal conductance matrix, the latter can be generated by

```
    ⎕←YRL←N DIFNET VR
¯85.46          93.44
 93.44         ¯54.03
```

Alternatively, the function *ITERNET* could be modified to return this matrix directly. The signal current excitation shown in Fig. 12.24 is described by

```
      ΔIR←.1 0
```

and therefore the signal voltage response is

```
      □←ΔVR←ΔIR⌹YRL
0.001313 0.002271
```

Thus, if the signal current is of 0.1 amp peak-to-peak amplitude, the signal voltages at nodes 1 and 2, with respect to node 3, will have peak-to-peak amplitudes of 1.313 millivolts and 2.271 millivolts respectively.

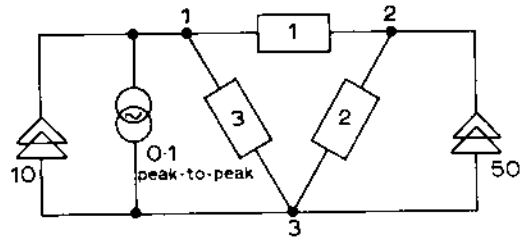


Fig. 12·24
The circuit of Fig. 11·6 with signal excitation

APPENDIX 1

A SAMPLE APL TERMINAL SESSION WITH ANNOTATION

Perusal of this session - or, better still, its repetition at an APL interactive terminal - may help the newcomer to gain sufficient understanding of the APL notation to be able to follow the foregoing text. This session should not be regarded as concise and extensive reference material on the APL notation: that is provided by Appendix 2.

What the user types is automatically indented. Results returned by the computer begin at the left margin. For the first two entries, activation of the carriage return key is indicated explicitly. Thereafter it should be understood that the key is activated at the end of each line typed by the user.

ACTUAL TERMINAL COPY	ANNOTATION
	SCALARS
4.5+6.3 carriage return 10.8 2×.5 carriage return 1 8÷¯16 ¯0.5 5-11 ¯6	Scalar addition, multiplication, division and subtraction. The symbol for subtraction (-) is different from the negative sign (¯) which denotes a negative value.
4=5 0 3≥1 1 16<.1 0	A logical proposition (=,≥,<, etc.) is either true (1) or false (0).
1∧1 1 0∨1 1 ~0 1	The logic functions AND, OR and NOT.

```
      3⌈6
6
      2*3
8
      *1
2.718281828
      ⍟2.718
1
      |¯4.2
4.2
      !3
6
```

Maximum of two numbers.

Raise to a power.

Exponential.

Natural logarithm (0 backspace *)

Magnitude.

Factorial (' backspace .)

```
      X←4
```

Assignment of a value to a named variable (computer returns no result; a definition is simply remembered).

```
      X
4
```

Interrogation of the value of *X*

```
      ⎕←X←7
7
```

An alternative means of defining and then displaying the value of a variable.

```
      X+23
30
```

Variables having values can be used as the arguments of APL functions.

```
      2×5+1
12
      (2×5)+1
11
```

APL expressions are executed from right to left, unless precedence is indicated by parentheses.

```
      2÷3
0.6666666667
      )DIGITS 3
WAS 10
      2÷3
0.667
```

Control of the number of significant digits to which all numerical values will subsequently be displayed.

VECTORS

```
      2 7 1+5 3 ¯2
7 10 ¯1
```

The primitive functions defined above are extended, element-by-element, to vectors.

```
      1  2 3× 4   5  6
4 10 18
      1 1 0 1∧ 1 0 1 1
1 0 0 1
```

One or more spaces serve to separate the elements of a vector.

```
      X←⍳5
      X
1 2 3 4 5
      ⎕←Y←2×⍳5
2 4 6 8 10
      X+Y
3 6 9 12 15
```

The "iota" function generates a vector of integers.

```
      +/X
15
      ∨/0 1 0 0 0
1
      ⌈/Y
10
```

Reduction of a vector to a scalar (the operation to the left of / is placed between each element of the vector appearing to the right of /).

```
      Y[2]
4
```

Indexing a vector.

```
      ⍴X
5
```

Dimension of a vector.

```
      2↑X
1 2
      S←¯1↑X
      S
5
       ¯2↓Y
2 4 6
```

Take the first two elements of the vector *X*.

Take the last element of *X* and call it *S*.

Drop the last two elements of *Y*

MATRICES

```
      M←3 3⍴⍳9
      M
 1 2 3
 4 5 6
 7 8 9
      ⎕←N←3 3⍴⍳5
 1 2 3
 4 5 1
 2 3 4
```

A 3-row, 3-column reshape (⍴) of the vector of integers from 1 to 9.

Vector to right of ⍴ is repeated sufficiently to form a matrix of the required dimension.

```
      M+N
  2  4  6
  8 10  7
  9 11 13
      M-N
 0 0 0
 0 0 5
 5 5 5
```

Primitive functions defined for scalars extend, element-by-element, to matrices.

```
      M[2;3]
6
      M[2;]
4 5 6
      M[;3]
3 6 9
```

Indexing a matrix to get an element, row or column.

```
      ρM
3 3
```

The dimension of a matrix.

```
      1 ¯1↓N
4 5
2 3
```

Drop first row and last column of *N*.

```
      ⍉M
 1 4 7
 2 5 8
 3 6 9
```

Matrix transpose (○ backspace \)

```
      M+.×N
 15 21 17
 36 51 41
 57 81 65
```

Ordinary matrix product. Operator +.× signifies element-by-element multiplication of rows of *M* with columns of *N*, followed by sum reduction over these products.

```
      M⌈.+N
  6  7  7
  9 10 10
 12 13 13
```

The same rule applies if other primitive functions replace + and ×in the operator +.× (GENERALIZED INNER PRODUCT)

```
      ⌹N
 ¯3.4   ¯.2   ¯2.6
  2.8    ¯.4  ¯2.2
  ¯.4    ¯.2    .6
```

Matrix inversion.

```
      X←10 20 30
      Y←(⌹N)+.×X
      Y
40 ¯30 10
      Y←X⌹N
      Y
40 ¯30 10
```

Solution of the simultaneous linear equation *X←N+.×Y*

Alternative expression.

```
      1 2 3∘.×1 2 3
 1 2 3
 2 4 6
 3 6 9
```

Outer product of two vectors, i.e., products of all pairs of elements. Vectors can be of unequal length.

```
      1 2 3∘.=1 2 3
 1 0 0
 0 1 0
 0 0 1
```

Same rule applies if another primitive function replaces × in the operator ∘.× (GENERALIZED OUTER PRODUCT)

HIGHER DIMENSIONAL ARRAYS

```
      3 2 4ρ⍳24
  1   2   3   4
  5   6   7   8

  9  10  11  12
 13  14  15  16

 17  18  19  20
 21  22  23  24
```

Definition of a 3-dimensional array. Vector to the left of ρ defines dimension of array.

FUNCTION DEFINITION

```
      ∇ R←UNIT N (carriage return)
[1]   R←(⍳N)∘.=⍳N ∇
```

Definition of a function, named *UNIT*, for the generation of a unit matrix of any size. After carriage return, computer types [1].

```
      UNIT 3
 1 0 0
 0 1 0
 0 0 1
      4×UNIT 2
4 0
0 4
```

The function *UNIT* can be used just like any other APL function.

```
      ∇ R←A BOX N
[1]   R←A×(⍳N)∘.=⍳N ∇
```

Defined functions can take one argument, on the right (as above), or two arguments.

```
      S←71 BOX 3
      S
 71  0  0
  0 71  0
  0  0 71
```

SPECIAL NOTE

Not intended as part of beginner's introduction to APL, but relevant to pages 170 and 183 of the text.

THE EXECUTE FUNCTION

```
      X←'ABCD GO'
      X
ABCD GO
      X[2 1 4 5 4 7 6]
BAD DOG
```

A character vector.

```
      N←3
      FN←'UNIT'
      ⍎FN,' N'
 1 0 0
 0 1 0
 0 0 1
```

The execute function (1 backspace ∘) takes a character vector as its argument and causes the associated expression to be executed.

APPENDIX 2

REFERENCE TO APL PRIMITIVE SCALAR AND MIXED FUNCTIONS

Monadic form *fB*		f	Dyadic form *AfB*	
Definition or example	Name		Name	Definition or example
`+B ↔ 0+B`	Plus	+	Plus	`2+3.2 ↔ 5.2`
`-B ↔ 0-B`	Negative	-	Minus	`2-3.2 ↔ ¯1.2`
`×B ↔ (B>0)-(B<0)`	Signum	×	Times	`2×3.2 ↔ 6.4`
`÷B ↔ 1÷B`	Reciprocal	÷	Divide	`2÷3.2 ↔ 0.625`
B | ⌈*B* | ⌊*B* 3.14 | 4 | 3 ¯3.14 | ¯3 | ¯4	Ceiling	⌈	Maximum	`3⌈7 ↔ 7`
	Floor	⌊	Minimum	`3⌊7 ↔ 3`
`*B ↔ (2.71828..)*B`	Exponential	*	Power	`2*3 ↔ 8`
`●*N ↔ N ↔ *●N`	Natural logarithm	●	Logarithm	`A●B ↔` Log *B* base *A* `A●B ↔ (●B)÷●A`
`|¯3.14 ↔ 3.14`	Magnitude	|	Residue	Case | *A*|*B* *A*≠0 | *B*-(|*A*)×⌊*B*÷|*A* *A*=0, *B*≥0 | *B* *A*=0, *B*<0 | Domain error

`!0 ↔ 1`
`!B ↔ B×!B-1`
or `!B ↔ Gamma(B+1)`
	Factorial	!	Binomial coefficient	`A!B ↔ (!B)÷(!A)×!B-A`
				`2!5 ↔ 10  3!5 ↔ 10`

`?B ↔` Random choice from `⍳B`	Roll	?	Deal	A Mixed Function (See Table 2)
`○B ↔ B×3.14159...`	Pi times	○	Circular	See Table at left
`~1 ↔ 0  ~0 ↔1`	Not	~		
		∧	And	
		∨	Or	
		⍲	Nand	
		⍱	Nor	
		<	Less	
		≤	Not greater	
		=	Equal	
		≥	Not less	
		>	Greater	
		≠	Not Equal	

A ∧ *B*	*A* ∨ *B*	*A* ⍲ *B*	*A* ⍱ *B*
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

Relations
Result is 1 if the relation holds, 0 if it does not:
3≤7 ↔ 1
7≤3 ↔ 0

(-*A*)○*B*	*A*	*A*○*B*
(1-*B**2)*.5	0	(1-*B**2)*.5
Arcsin *B*	1	Sine *B*
Arccos *B*	2	Cosine *B*
Arctan *B*	3	Tangent *B*
(¯1+*B**2)*.5	4	(1+*B**2)*.5
Arcsinh *B*	5	Sinh *B*
Arccosh *B*	6	Cosh *B*
Arctanh *B*	7	Tanh *B*

Table of Dyadic ○ Functions

Name	Sign[1]	Definition or example[2]
Size	ρA	$\rho P \leftrightarrow 4$ $\rho E \leftrightarrow 3\ 4$ $\rho 5 \leftrightarrow \iota 0$
Reshape	$V\rho A$	Reshape A to dimension V $3\ 4\rho\iota 12 \leftrightarrow E$ $12\rho E \leftrightarrow \iota 12$ $0\rho E \leftrightarrow \iota 0$
Ravel	$,A$	$,A \leftrightarrow (\times/\rho A)\rho A$ $,E \leftrightarrow \iota 12$ $\rho,5 \leftrightarrow 1$
Catenate	V,V	$P,\iota 2 \leftrightarrow 2\ 3\ 5\ 7\ 1\ 2$ $'T','HIS' \leftrightarrow 'THIS'$
Index[3,4]	$V[A]$	$P[2] \leftrightarrow 3$ $P[4\ 3\ 2\ 1] \leftrightarrow 7\ 5\ 3\ 2$
	$M[A;A]$	$E[1\ 3;3\ 2\ 1] \leftrightarrow \begin{matrix} 3 & 2 & 1 \\ 11 & 10 & 9 \end{matrix}$
	$A[A;..;A]$	$E[1;] \leftrightarrow 1\ 2\ 3\ 4$ $E[;1] \leftrightarrow 1\ 5\ 9$ $'ABCDEFGHIJKL'[E] \leftrightarrow \begin{matrix} ABCD \\ EFGH \\ IJKL \end{matrix}$
Index generator[3]	ιS	First S integers $\iota 4 \leftrightarrow 1\ 2\ 3\ 4$ $\iota 0 \leftrightarrow$ an empty vector
Index of[3]	$V\iota A$	Least index of A in V, or $1+\rho V$ $P\iota 3 \leftrightarrow 2$ $4\ 4\iota 4 \leftrightarrow 1$ $5\ 1\ 2\ 5\iota E \leftrightarrow 3\ 5\ 4\ 5$ $5\ 5\ 5\ 5$
Take	$V\uparrow A$	Take or drop $\lvert V[I]$ first ($V[I]\geq 0$) or last ($V[I]<0$) elements of coordinate I $2\ 3\uparrow X \leftrightarrow \begin{matrix} ABC \\ EFG \end{matrix}$ $^-2\uparrow P \leftrightarrow 5\ 7$
Drop	$V\downarrow A$	
Grade up[3,5]	ΔA	The permutation which would order A (ascending or descending) $\Delta 3\ 5\ 3\ 2 \leftrightarrow 4\ 1\ 3\ 2$
Grade down[3,5]	∇A	$\nabla 3\ 5\ 3\ 2 \leftrightarrow 2\ 1\ 3\ 4$
Compress[5]	V/A	$1\ 0\ 1\ 0/P \leftrightarrow 2\ 5$ $1\ 0\ 1\ 0/E \leftrightarrow \begin{matrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{matrix}$ $1\ 0\ 1/[1]E \leftrightarrow \begin{matrix} 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{matrix} \leftrightarrow 1\ 0\ 1/E$
Expand[5]	$V\backslash A$	$1\ 0\ 1\backslash\iota 2 \leftrightarrow 1\ 0\ 2$ $1\ 0\ 1\ 1\ 1\backslash X \leftrightarrow \begin{matrix} A & BCD \\ E & FGH \\ I & JKL \end{matrix}$
Reverse[5]	ϕA	$\phi X \leftrightarrow \begin{matrix} DCBA \\ HGFE \\ LKJI \end{matrix}$ $\phi P \leftrightarrow 7\ 5\ 3\ 2$ $\phi[1]X \leftrightarrow \ominus X \leftrightarrow \begin{matrix} IJKL \\ EFGH \\ ABCD \end{matrix}$
Rotate[5]	$A\phi A$	$3\phi P \leftrightarrow 7\ 2\ 3\ 5 \leftrightarrow {}^-1\phi P$ $1\ 0\ {}^-1\phi X \leftrightarrow \begin{matrix} BCDA \\ EFGH \\ LIJK \end{matrix}$
Transpose	$V\lozenge A$	Coordinate I of A becomes coordinate $V[I]$ of result $2\ 1\lozenge X \leftrightarrow \begin{matrix} AEI \\ BFJ \\ CGK \\ DHL \end{matrix}$ $1\ 1\lozenge E \leftrightarrow 1\ 6\ 11$
	$\lozenge A$	Transpose last two coordinates $\lozenge E \leftrightarrow 2\ 1\lozenge E$
Membership	$A\epsilon A$	$\rho W\epsilon Y \leftrightarrow \rho W$ $P\epsilon\iota 4 \leftrightarrow 1\ 1\ 0\ 0$ $E\epsilon P \leftrightarrow \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$
Decode	$V\perp V$	$10\perp 1\ 7\ 7\ 6 \leftrightarrow 1776$ $24\ 60\ 60\perp 1\ 2\ 3 \leftrightarrow 3723$
Encode	$V\top S$	$24\ 60\ 60\top 3723 \leftrightarrow 1\ 2\ 3$ $60\ 60\top 3723 \leftrightarrow 2\ 3$
Deal[3]	$S?S$	$W?Y \leftrightarrow$ Random deal of W elements from ιY

1. Restrictions on argument ranks are indicated by: *S* for scalar, *V* for vector, *M* for matrix, *A* for Any. Except as the first argument of *S*⍳*A* or *S*[*A*], a scalar may be used instead of a vector. A one-element array may replace any scalar.

2. Arrays used in examples: *P* ←→ 2 3 5 7 *E* ←→ 1 2 3 4 / 5 6 7 8 / 9 10 11 12 *X* ←→ *ABCD* / *EFGH* / *IJKL*

3. Function depends on index origin.

4. Elision of any index selects all along that coordinate.

5. The function is applied along the last coordinate; the symbols ⌿, ⍀, and ⊖ are equivalent to /, \, and ⌽, respectively, except that the function is applied along the first coordinate. If [*S*] appears after any of the symbols, the relevant coordinate is determined by the scalar *S*.

Notes to Table of Mixed Functions *

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* Reprinted from the APL\360 User's Manual, IBM Corporation, 1968.

EXERCISES

The following pages present a selection of exercises which support the material discussed in the text. A few exercises also serve to introduce new concepts and devices for the first time. The substitution theorem, Tellegen's theorem, the operational amplifier, the field-effect transistor, the gyrator and the voltage-controlled current source are topics which are first introduced via an exercise.

In the sense that a good set of exercises is developed only by refinement over a period of time, the exercises offered here must be regarded as a starting point. With a text that is intended to be supported by work at an APL terminal, however, there is an additional problem. While exercises at a terminal can be extremely valuable, I feel it is necessary also to develop in the student an ability to analyse circuits informally and intuitively, often at the same time making judicious engineering approximations. It is for this reason that some exercises - whether too many or too few I cannot yet say - do not require recourse to an APL terminal.

The exercises vary in difficulty. Most should be capable of completion in about fifteen minutes or less. Others may take a little longer, while a few may turn out to be too extensive for the freshman or may form suitable subjects for a miniproject form of activity.

Where the completion of one exercise is a prerequisite for the consideration of another, the number of the prerequisite exercise is shown in parentheses before the exercise number. For some exercises, certain named APL functions (often preferably locked) should be made available to the student. The definition of these functions appears in the FUNCTION LISTINGS section. The teacher should ensure that these functions can be copied into the student's workspace at an APL terminal.

EXERCISES: CHAPTER 2

1. Simulate a single measurement of the current response of a component *C1* to a voltage of 4 volts by assigning a value of 4 to *V* and employing *V* as the argument of the (locked) function *C1*. Now define an excitation vector *V* whose elements are a random selection of integers between 1 and 100. Compute the current response *I*. Plot *I* versus *V*, and from examination of the result propose a mathematical model of the function *C1*. What single parameter defines this function ?

(1) 2. Use the vectors *I* and *V* generated in exercise 1 in turn as the arguments of *DIFF*. Now enter *(DIFF I)÷DIFF V*. Relate the results of these applications of the function *DIFF* to the mathematical model proposed in exercise 1.

3. Employ as vectors of voltage excitation *V* for the component *C1* the vectors

```
2× ⍳10          ⍳7          ¯6+⍳11
```

and generate the corresponding vectors *I* of current response. Investigate *DIFF I* and *(DIFF I)÷DIFF V* in each case, and relate the results to a sketch of the function *C1*.

(1) 4. Define an APL function *MODEL* which embodies the mathematical model proposed for the function *C1*. Test your model in two ways. First, by applying the same vector of voltage excitation to both *MODEL* and *C1* and examining the current response. Second, by entering the expressions

```
0=(C1 V)-MODEL V
∧/0=(C1 V)-MODEL V
```

where *V* is an arbitrarily chosen vector of voltage excitation.

5. Assume you have meters reading to two significant figures by entering *)DIGITS 2*. Measure the component described by the function *C2* which takes a voltage argument, and propose a model which is linear and homogeneous. Test the model against the function *C2* (as in exercise 4) by defining an APL

function incorporating the model. If there is any discrepancy, determine the accuracy within which the model is valid for the range of voltage excitation you have employed. Compute both actual and fractional deviations of the model from the function *C2*. Devise a simple APL expression involving *C2* and the function incorporating the model, and which returns 1 or 0 according to whether the current responses of the model and the component agree or not to within a specified error.

6. By means of exploratory measurements on the component *C3*, each time employing a single (scalar) value of excitation voltage, determine the voltage response to a current excitation of 5 amps.

(1) 7. From examination of the plot generated in exercise 1, deduce the relation which is the inverse of *C1*, and which expresses voltage as a function of current. Incorporate this relation in an APL function *INVERSEC1*, and test to see that it is the inverse of *C1*.

8. Measure the component *CV1* (the *V* in the function name indicates that voltage is the argument) and propose a linear and homogeneous model which is accurate around the origin. Consider the vector of voltage excitation `V←¯11+⍳21`. Determine APL expressions whose result displays (a) those elements of *V* for which the discrepancy in response current between the model and *CV1* exceeds 1 ampere, and (b) the total range of *V* for which the model is accurate to within 1 ampere.

9. A component is modelled by the APL function *INVC1*, which takes a current argument. Carry out sufficient measurements of voltage response to enable you to propose a mathematical model of the function. Incorporate this model in an APL function and test it. It should be the inverse of the function *C1*, first encountered in exercise 1. Test this.

10. Determine the value of resistance which characterises the component *CI1*, which takes a current argument. Create a model of *CI1*, and test it.

(10) 11. A voltage excitation of 10 volts is applied to the component *CI1*. What is the current response ?

12. A component modelled by the function *CV2* will be used in a circuit in such a way that its voltage will always lie between 10 and 12 volts. Propose a linear

model of the component which is a good approximation over this range, and which will generallv be easier to handle than the nonlinear relation embodied in *CV2*. Generate superimposed plots of *CV2* and your model for a range of voltage between 5 and 13 volts.

13. A component *CV3* will be used in such a way that its voltage *V* has an average value of 3.5 volts, and never exceeds the limits of 3 and 4 volts. What is of interest to the circuit designer is the relation between changes ΔI and ΔV in *I* and *V* around their average values. Propose a linear and homogeneous expression which relates these changes, and test it.

14. Carry out measurements on the component *CV4* with the object of modelling the component by a polynomial expression. Define an APL function incorporating the model you propose, and test it.

15. Carry out measurements on the component *CV5*, and define an APL function which is an accurate model over the range of excitation voltage from ¯20 to 20 volts. How accurate is the model ?

16. Determine the incremental conductance of the component *CV6* for an excitation voltage of 3 volts. Repeat for excitation voltages of 1, 2 and 4 volts.

(16) 17. Over what excursions in voltage from the value of 3 volts does the appropriate incremental conductance determined in exercise 16 predict current response that is accurate to within (a) 10% of the actual current, (b) 0.1 ampere and (c) 10% of the current excursion ?

18. Determine the incremental conductance of the component *CV7* for excitation voltages of ¯10, ¯5, 0, 5 and 10 volts. What do you notice about the result ? Can the conclusion be drawn that if the incremental conductance is constant, then the component is both linear and homogeneous ?

(18) 19. Test the conclusion drawn in exercise 18 on the component *CV8*.

20. Sketch, on graph paper, the function *I*←*V*⋆3 describing the voltage-current relation of a particular component. By graphical means, determine an approximate value for the incremental conductance for voltages of 1, 2 and 3 volts.

21. Determine the incremental resistance of the component *CV9* for an excitation current of 15 amperes. What is the incremental conductance for this value of current ? Test your answer.

22. Determine the incremental conductance of the component *CV*1 for *V* equal to 0.5, 1.5, 2.5, 3.5 and 4.5 volts, by simple use of the function *DIFF*. Now calculate the current response as *V* increases in 1 volt steps from 0 to 5 volts, using only knowledge of the incremental conductance appropriate to each step. Compare the final current response with the value *CV*1 5. Give the reason for the agreement or lack of it. Now determine the incremental conductances more accurately and repeat the exercise.

23. A semiconductor diode is modelled by the function *SEMIV*1, which takes a voltage argument. Plot measurements of its current-voltage characteristic for the range of voltage excitation ¯0.5<*V*<0.5. At what forward voltage does the diode apparently begin to conduct a substantial current ? Repeat the measurement and plot for other ranges of excitation voltage.

24. Obtain a ploynomial expression relating the current and voltage of the component *SEMIV*1 to a good accuracy for voltages between the limits of ¯40 and 40 millivolts. What is the approximate maximum current error in this range ? Obtain a plot of the polynomial expression over a much wider range of voltage.

25. Define an APL function *DIV* which computes the quotient of successive elements of its vector argument.

(25) 26. By use of the function *DIV*, test the following components to see if they apparently lend themselves to description by an expression in which an exponential term is dominant:

*CV*10 *CI*2 *CI*3

(25) 27. Generate the current response *I* of the component *SEMIV*1 to the sequence of voltage excitations *V*←0.05+0.1×¯11+⍳21. Now employ *I* as the argument of *DIV*. From the result, determine the value of the parameter *A* in the relation *I*←*IS*×¯1+**A*×*V* embodied in the function *SEMIV*1.

28. Generate the current response of the component *DIODEV* to the sequence of voltage excitations *V*←0.05+0.1×¯11+⍳21. Where possible, generate the natural logarithm of *I*, and plot it versus *V*. Propose a model for the component *DIODEV*, incorporate it within an APL function, and test it.

29. Figure E.2.1 shows a plot derived from some measurements on a semiconductor diode. From examination of this plot, propose a mathematical model of the diode. Incorporate this model in an APL function, and reproduce as much of the figure as possible.

30. Determine the local conductance of the component *DIODEV* for each voltage excitation described by the vector

V←0.05+0.1×¯11+⍳21 by simple application of the function *DIFF*. Name the vector of local conductances *GG* (it should be a 20-element vector). Now generate the vector of voltages *VQ* for which each element of the vector *GG* is relevant, as well as the corresponding vector *IQ*←*DIODEV VQ* of currents. Experiment with the vectors *GG*, *VQ* and *IQ* to seek a very simple relation between *GG* and either *VQ* or *IQ* that is valid over a substantial fraction of the range of voltage excitation. Explain your result by reference to a mathematical model of the function *DIODEV*.

31. An ideal diode can be described by the relation *I*←0 for *V*<0 ; *V*←0 for *I*>0. Sometimes, an actual diode can be satisfactorily modelled by the piecewise-linear relation *I*←0 for *V*<*VC* ; *I*←*G*×*V*-*VC* for *V*>*VC*. Sketch the general form of these two diode descriptions. Devise a piecewise-linear model for the component *DIODEV*, assuming that the component voltage will lie within the range ¯0.5<*V*<0.5. Incorporate this model in an APL function, and see how good it is.

32. For the component described by equation 2.6 on page 22, are there any values of *V* for which a value of *I* does not exist ? Are there any values of *I* for which a value of *V* does not exist ? In each case, if the answer is "yes", give an example. Sketch voltage-current curves for which the answers to the above pair of questions is (yes, yes), (yes, no), (no, yes) and (no, no).

33. Determine the voltage response of the component *COMV*1 to an excitation current of 5 amperes by first defining an APL function which is the inverse of *COMV*1.

(25) 34. Use the function *DIV* in an exploratory manner, and then either of the functions *TRP* or *TRM*, to determine the mathematical model of the component *COMV*2

35. Carry out measurements upon one or more of the following components

*CV*11 *CI*4 *CI*5 *CV*12

Incorporate each model in an APL function, and test it. In each case state the approximate maximum current or voltage response error for a stated range of voltage or current excitation, respectively.

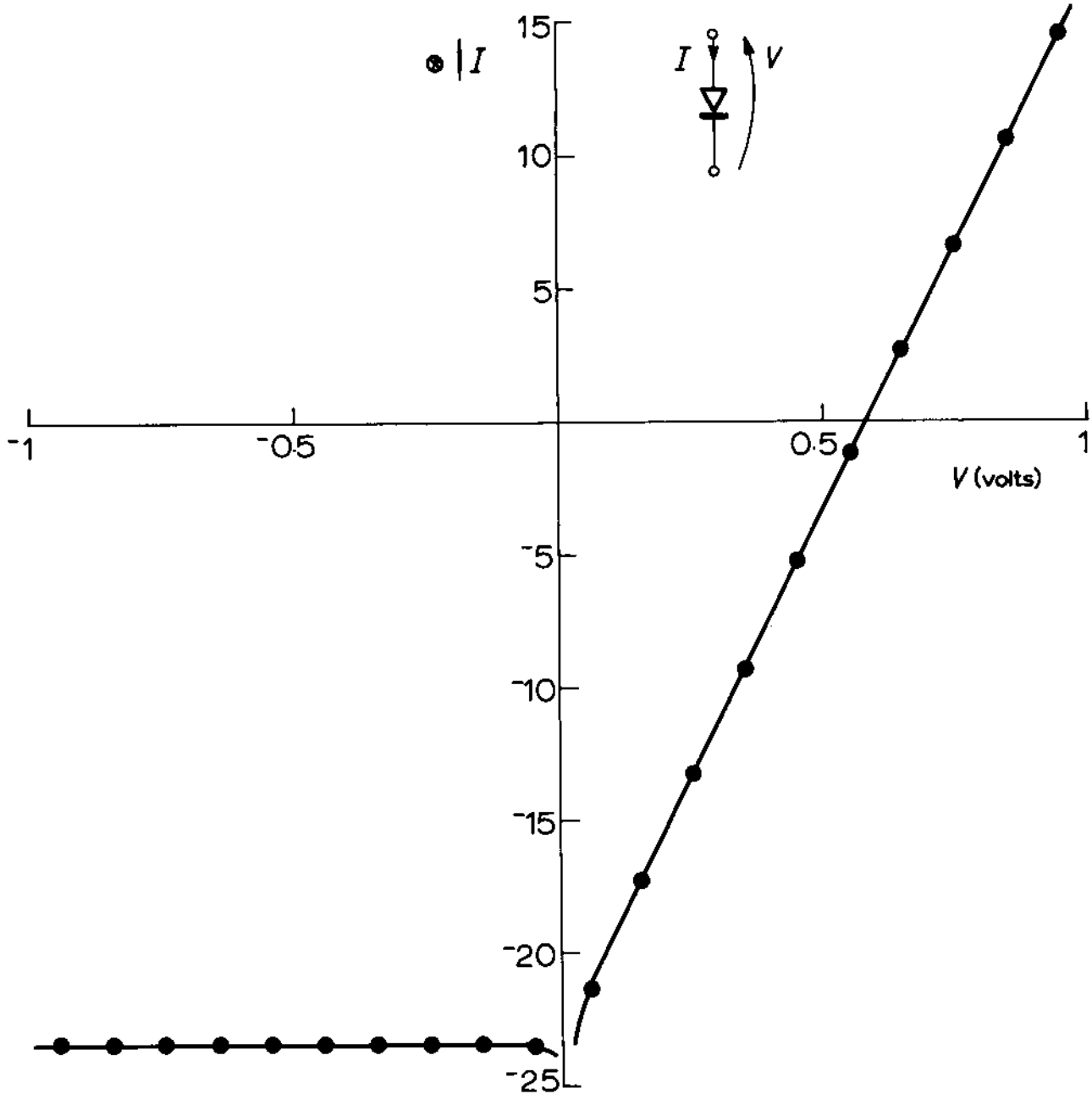


Fig. E. 2.1.

EXERCISES: CHAPTER 3

1. Draw a circuit containing at least six components, each represented by a simple box, as in Fig.3.1. Choose a set of component currents that satisfies Kirchhoff's current law. Now change one of the component currents to a new value, and determine a new set of currents obeying Kirchhoff's current law, but keeping as many component currents as possible at their previous value.

2. If the currents entering every junction point except one have been checked and found to sum to zero, is it necessary to carry out the same check on the currents entering the last junction point ? Why ?

(1) 3. Assign the name *IB* to the first set of component currents chosen in exercise 1. Index *IB* to yield the currents entering junction 2. From examination of the result, and without reference to the circuit diagram, can you tell which currents are entering junction 2 and which are leaving ? Use the result of the indexing to calculate the total current entering junction 2.

4. Draw a circuit containing at least four junctions (use the circuit of exercise 1 if it satisfies this requirement) and define the incidence vector *A*3 for junction 3: the elements of this vector are 1, ¯1 or 0 according to whether the reference currents of the numbered components leave, enter or are not incident upon this junction, respectively. Now define a vector *IB* of component currents which satisfy Kirchhoff's current law. Carry out the following calculations:

```
A3×IB        +/A3×IB        A3+.×IB
```

What is computed by each of the above calculations ? If the elements 1 and ¯1 in *A*3 were interchanged, what change would you expect in each of the above calculations ? Test your answer.

5. Draw a circuit containing five junctions and arbitrarily assign a potential to each junction (e.g., *P*←¯51+5?101). Now define a vector *B*5 associated with component number 5, and whose elements are 1, ¯1 or 0 respectively according to whether the numbered junctions are at the positive or negative reference point of the component, or not associated with the component at all. Compute the voltage of component number 5, using the vectors *IB* and *P*. Junction 3 is now chosen as the

reference point for junction potentials: compute the new potential of each junction (i.e., with respect to this new reference) and check that the voltage of component number 5 is the same as before by carrying out the same calculation with *B*5 and *P*.

6. Define a dyadic (i.e., two-argument) function *COMVOLT* satisfying the following requirement. Its left-hand argument is a 2-element vector whose elements are the positive and negative voltage reference junctions of a component, its right-hand argument is the vector of junction potentials with respect to some arbitrarily chosen reference, and its result is the component voltage. Test your function for a circuit in which component 3 has junctions 2 and 4 as its positive and negative reference junctions, by entering the following expressions:

```
2 4 COMVOLT P
2 4 COMVOLT 10+P
4 2 COMVOLT P
2 4 COMVOLT 5×P
4 2 COMVOLT (-P[2])+P
```

7. For a circuit containing five junctions arbitrarily assign junction potentials and hence determine the component voltages. For at least four different loops, verify that the sum of the voltages associated with components that form a closed loop is zero, but only provided that reference directions are observed.

8. Define a function *GPP*which returns the equivalent conductance of the parallel connection of two conductances, the values of which are the arguments of *GPP*. Test the function. Use it to determine the current *I* in the circuit of Fig.E.3.1 when the battery voltage *V* is adjusted to be 5 volts. Determine the separate currents which flow in the two conductances, and check that they and the current *I* obey Kirchhoff's current law.

9. Define a dyadic function *ISPLITR* which determines the fraction of the total current flowing in each of two resistances connected in parallel, the values of which are the arguments of *ISPLITR*. Define another function *ISPLITG* which carries out the same calculation when its arguments are the conductances of the components connected in parallel.

10. For the circuit of Fig.E.3.2 determine the value of the current *I* when the voltage *V* of the battery is 10 volts. Take careful note of the reference directions of *I* and*V*.

11. The circuit of Fig.E.3.3 contains a (nonlinear) exponential diode described by the function *DV*. Determine

the voltage across the diode to an accuracy of 3 significant digits.

12. Table E.3.1 lists the measurements of voltage and current carried out on a certain diode: knowledge of the physical mechanism of the diode suggests a model of the form shown in Fig.3.17(f). Find the values of the components of this model for the measured diode.

Voltage (volts)	Current (amps)
-1.00E0	-1.00E-5
-9.00E-1	-9.00E-6
-8.00E-1	-8.00E-6
-7.00E-1	-7.00E-6
-6.00E-1	-6.00E-6
-5.00E-1	-5.00E-6
-4.00E-1	-4.00E-6
-3.00E-1	-3.00E-6
-2.00E-1	-2.00E-6
-1.00E-1	-1.00E-6
0.00E0	0.00E0
1.00E-1	1.00E-6
2.00E-1	2.09E-6
3.00E-1	6.51E-6
4.01E-1	1.46E-4
5.57F-1	5.74E-3
2.92E0	2.32E-1
9.46E1	9.39E0

Table E.3.1

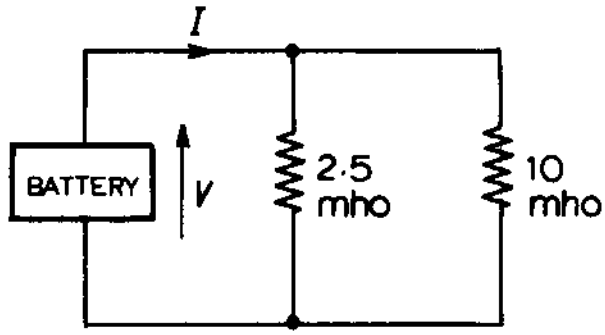


Fig. E. 3·1

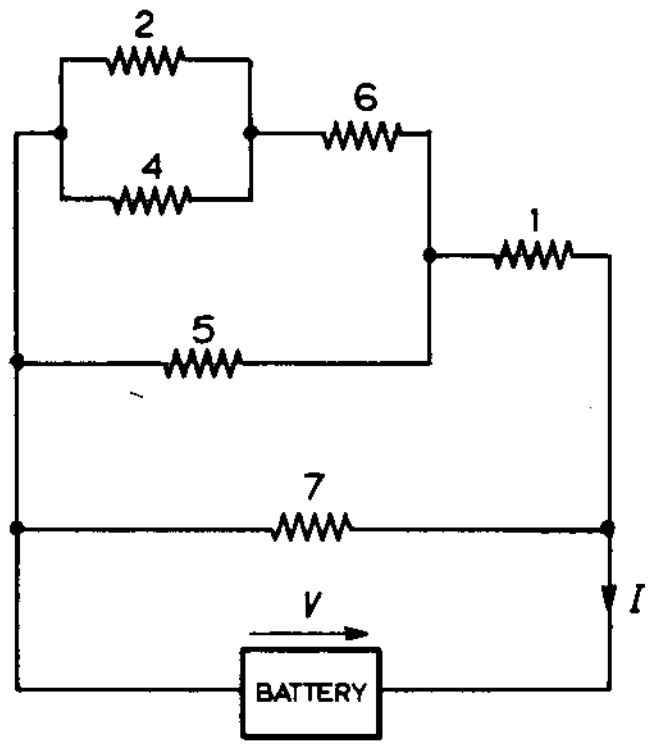


Fig. E. 3·2

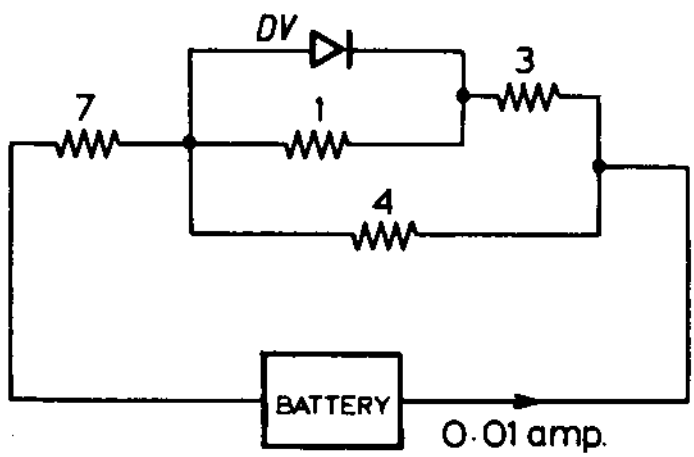


Fig. E. 3·3

EXERCISES: CHAPTER 4

1. By connecting different values of resistance as a load on a 2-terminal component, the values of load current given in Table E.4.1 were obtained. Propose a Thevenin equivalent circuit for the 2-terminal component, under the assumption that the measured load currents flow out of the component terminal labelled A, and into the terminal labelled B.

Resistance (ohms)	Current (amps)
0	2.000
1	1.667
2	1.429
3	1.250
4	1.111
5	1.000

Table E.4.1

2. The function *CR*1 describes a 2-terminal component in that if its single argument is the value of the resistance connected to the component, then the function returns the component voltage. By means of simulated measurements using the function *CR*1 determine the Thevenin equivalent circuit of the component. Obtain a plot of component voltage versus load resistance. At what value of resistance load is the component voltage equal to half its open-circuit value ? Is this resistance value consistent with the value you have proposed for the internal resistance of the Thevenin equivalent circuit ?

3. Obtain the Thevenin equivalent circuit of a component whose current is returned by the monadic function *CR*2 which takes as its argument the value of the resistance load connected to the component. Define a new function describing the same component, which returns current as a function of voltage. Test this function against *CR*2.

4. A component is simulated by the function *CR*3 which takes resistance load as its argument and returns the value of the component current. By means of simulated measurements, deduce the Norton equivalent circuit of the

component.

5. If a component can be modelled by a Thevenin equivalent circuit, then it can also be modelled by a Norton equivalent circuit. How are the parameters (*VOC*, *RIN*), (*ISC*, *GIN*) of the two alternative models related ?

6. In Fig. E.4.1 two separate components are seen to be excited by the same voltage source. The current-voltage relations of the two components are described by the functions *COM*1 and *COM*2, each of which takes a voltage excitation argument and returns the current response. Determine, for each component, the power supplied to the component. If the polarity of each voltage source is reversed, will the same power as before be supplied to each component ? Test your answer. What type of component is modelled by *COM*1 and *COM*2 ?

7. The APL functions *COMV*3, *COMI*1 and *COMV*4 simulate the current-voltage relations of three 2-terminal components: an *I* or *V* in the function name indicates whether current or voltage, respectively, is the argument. By means of simulated measurements, determine if each component is active or passive.

8. The function *PSUP* describes the supply of power to a resistance load by a certain component. The single argument of *PSUP* is the value of the resistance load. What is returned by the function is the power supplied by the component to the load. By experimenting with the function *PSUP* determine the Thevenin and Norton equivalent circuits of the component and the maximum available power associated with the component. Draw a diagram showing how measurements would be made in the laboratory and which would enable the power supplied to the resistance load to be calculated.

9. Fig. E.4.2 shows a number of interconnections of resistances and ideal sources. In each case state

   (a) whether the connection is permissible,
   (b) if the answer to (a) is 'No', the reason why,
   (c) if the answer to (a) is 'Yes', the value of the variable whose symbol is shown (*I*,*V* and *P* denote current, voltage and power received, respectively).

   In each case try to obtain the answer in as few intermediate steps as possible, by ignoring irrelevant information.

10. Two components are described by the functions *FFV*1 and *FFV*2:

```
      ∇I←FFV1 V                    ∇I←FFV2 V
[1] I←¯10+2×V ∇              [1] I←0.1×V*3 ∇
```

    The parallel combination of these components is excited by a current source of 2 amps. What is the voltage across

each component ? Check your answer by testing to see whether the currents in the circuit obey Kirchhoff's current law. Now suppose that the components are connected together to form a simple loop. What current flows in the loop ? Check your answer against Kirchhoff's voltage law.

11. The diode in the circuit of Fig. E.4.3 is described by the current-voltage relation *I←V*3*. By means of a load-line construction, determine the current in the diode when connected to the component whose Thevenin equivalent circuit parameters have the values shown.

12. Define a dyadic APL function *PASSIVITY* which can test to see if a 2-terminal component is active or passive. Its left-hand argument should be the excitation vector for which passivity is to be explored, while its right-hand argument is a character string which is the name of the function simulating the 2-terminal component. Determine whether the components *ONEBOXI*1 and *BATTERYV*1 are active or passive over the excitation range indicated below by entering:

```
( ⍳5 ) PASSIVITY 'ONEBOXI1'
(¯11+⍳21 ) PASSIVITY 'BATTERYV1'
```

Note: it is likely that you will have to employ the execute function (⍎) within the function *PASSIVITY* (see Appendix 1).

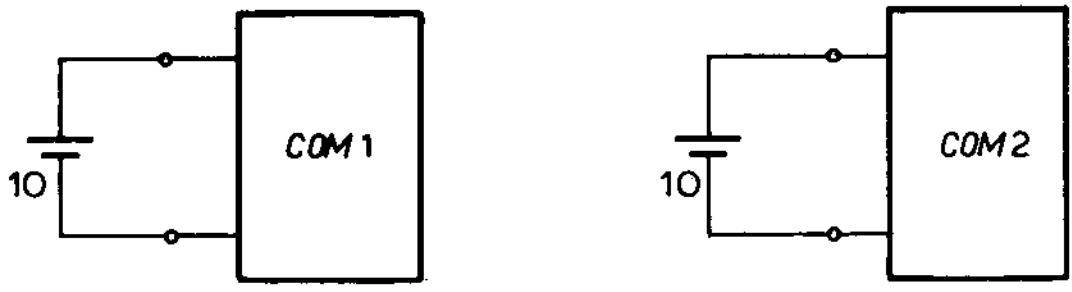
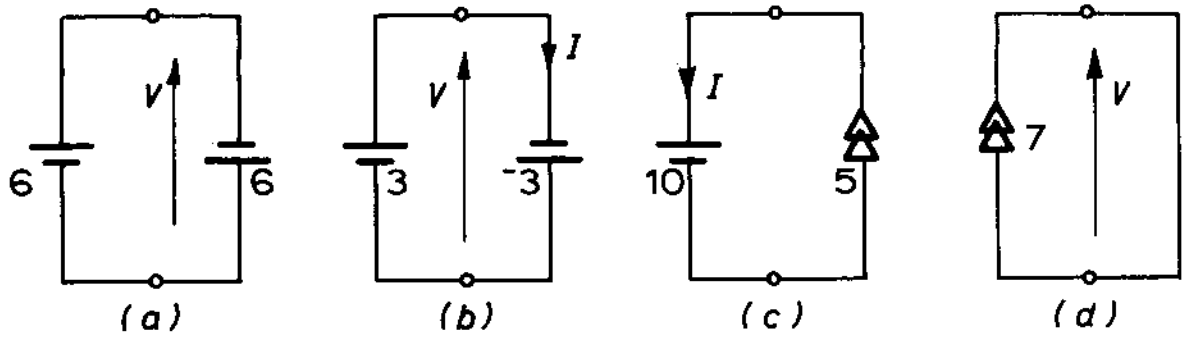
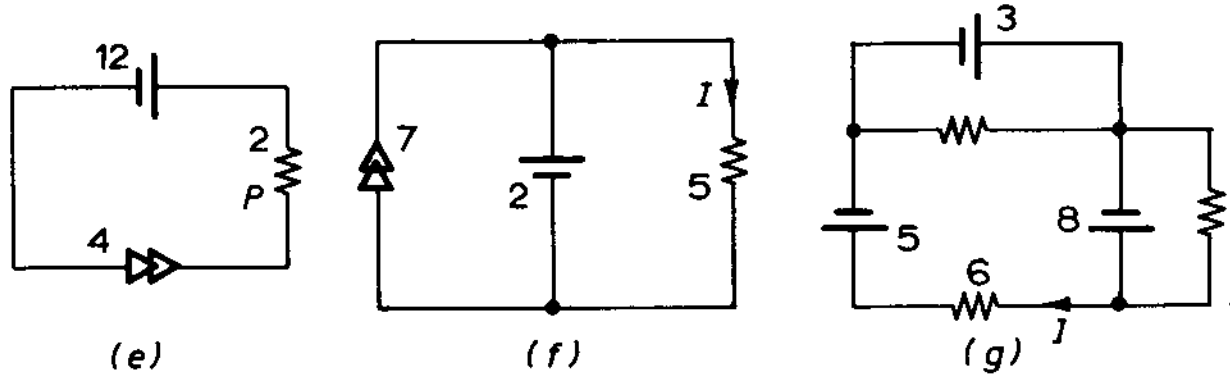


Fig. E.4.1





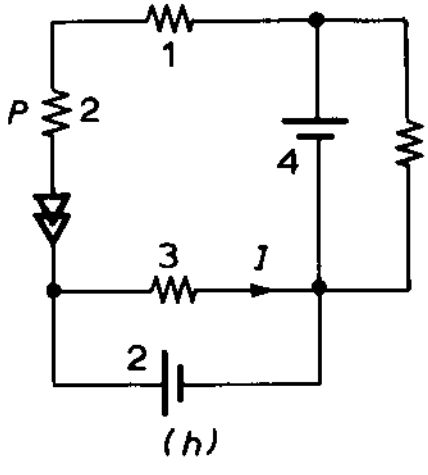


Fig. E. 4.2

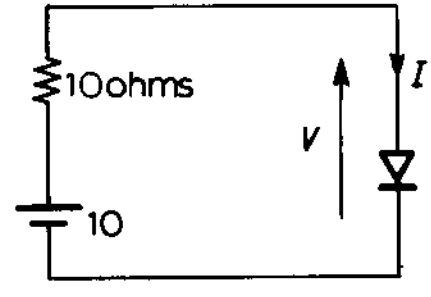


Fig. E. 4.3

EXERCISES: CHAPTER 5

1. A 2-terminal component is simulated by the function *ONEV20*, which takes a voltage argument. The component is to be subjected to a range of voltage excitation in which the constant component can assume integer values between 1 and 5 volts and, at the same time, the (square-wave) time-varying component can have peak-to-peak amplitudes of .1, .2, .3 and .4 volts. Carry out simulated measurements, and use the function *DIFF*, to find the local conductance of the component for quiescent voltages of 2 and 4 volts. Is the component nonlinear, or linear and homogeneous ? Define a model of the component which is accurate at a chosen quiescent point for time-varying excitation of infinitely small amplitude. For what amplitude of time-varying voltage excitation will this model predict the amplitude of the time-varying current response to an accuracy within plus and minus 10 per cent ?

2. Two linear 2-terminal components are separately connected to a resistance of value *R*. As *R* is varied, the average value *VA* of the voltage across *R* is noted, as is the peak-to-peak amplitude *VPP* of this voltage. The results are shown in the table below. Derive a Thevenin equivalent circuit of the form of Fig.5.4 (page 63) for each of the two components. Could any of the entries in the table have been omitted without preventing you from finding the correct equivalent circuit ?

COMPONENT A			COMPONENT B		
R ohms	*VA* volts	*VPP* volts	*R* ohms	*VA* volts	*VPP* volts
10	6	0.6	10	30	6.0
2	2	0.2	2	22	4.4

3. By exploiting your knowledge of the equivalent resistance of linear homogeneous components connected in series or parallel, and by invoking the superposition principle, determine the waveform of the current *I* in the circuit of Fig. E.5.1.

4. For the circuit of Fig. E.5.2, determine the peak-to-peak amplitude of the current *I*. Show how this calculation can be associated with a circuit containing a single conductance and a source of signal excitation.

5. A circuit containing both constant and time-varying sources of excitation is shown in Fig. E.5.3. Find the constant and time-varying components of the current *I*. Now suppose that the one ohm resistance connected directly between terminals A and B is replaced by a component whose current-voltage relation is *I*←*V**3, so that all average currents and voltages remain unchanged (check this). Can it be assumed that the time-varying component of the current *I* will also remain unchanged ? Test your answer.

6. Usually the amplitude of a time-varying signal applied to a nonlinear component is kept small in order to avoid distortion. Sometimes, however, distortion is intentional, as when an easily generated triangular voltage waveform is applied to a nonlinear component to generate a current waveform which approximates to a sinusoid (it is difficult to generate a very low frequency sinusoid directly). Experiment with a triangular waveform of voltage applied to a component described by *I*←*V**3, and see how closely the current waveform approximates to a sinusoid.

(2) 7. The same measurements described in exercise 2 were applied to a third component, with the results shown below. Is it possible to obtain a linear Thevenin equivalent circuit of the form shown in Fig. 5.4 (page 63) ? If so, obtain its parameters. If not, say why.

R ohms	*VA* volts	*VPP* volts
10	40	3
2	4	0.4

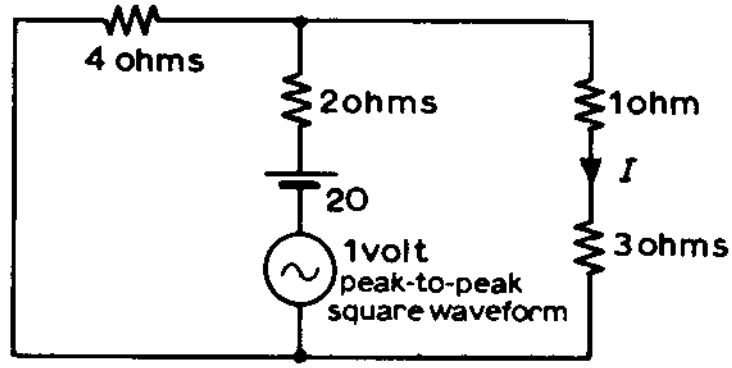


Fig. E.5.1

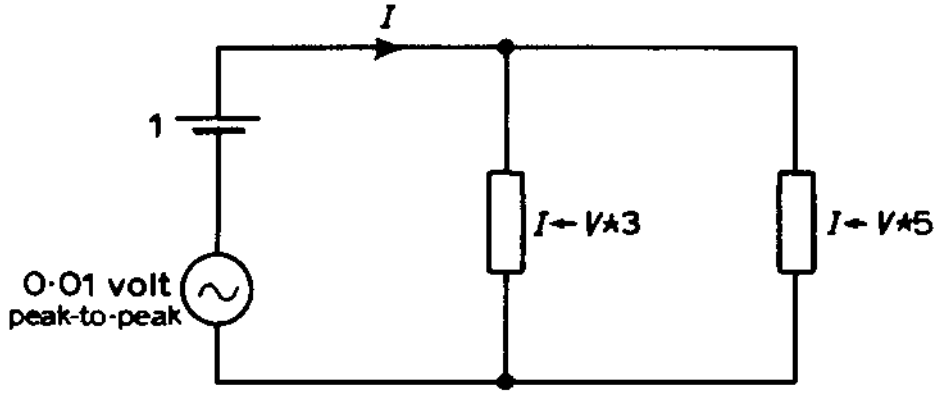


Fig. E.5.2

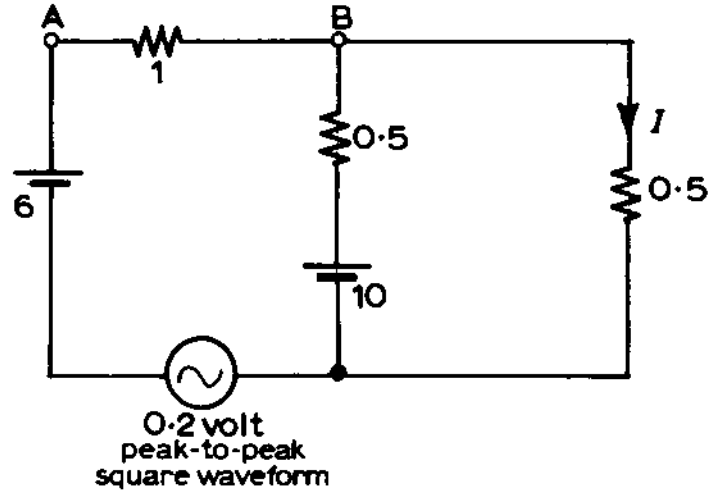


Fig. E.5.3

EXERCISES: CHAPTER 6

1. Fig. E.6.1 shows a 3-terminal component with current excitation. Draw the same component with excitation as shown in Fig.6.8, but which is equivalent to the excitation shown in Fig. E.6.1.

2. Deduce the current entering each terminal, and the voltage between each terminal, of the component shown in Fig. E.6.2 : observed currents and voltages are indicated by values. Be sure to indicate the reference direction of each current and voltage.

3. Refer to Fig. E.6.3. If a 3-terminal component is described by the function *COMV*5 whose argument is a 2-element vector *V* and which returns a 2-element vector *I*, what is the value of *I*[1] and *I*[2] if *V*[1]=4 and *V*[2]=¯2 ? What current flows into the terminal whose current is not named ?

4. For the 3-terminal component simulated by the function *COMV*5, simulate a series of measurements in which *V*[1] is held constant at 6 volts and *V*[2] is varied in equal steps of 2 volts from 4 volts to 10 volts inclusive. What are the values of the corresponding steps in the currents *I*[1] and *I*[2] ?

(4) 5. On the basis of the measurements carried out in exercise 4, determine any of the conductance parameters that can be calculated.

(4) 6. Under the assumption that the function *COMV*5 is linear and homogeneous determine, on the basis of the measurements carried out in exercise 4, any of the conductance parameters that can be calculated. Could any parameter be calculated on the basis of a single measurement ? (i.e., from knowledge of the 2-element vector of response current associated with a given 2-element vector of voltage excitation).

7. Fig. E.6.4 refers to a laboratory experiment in which measurements were made of the voltages and currents of a 3-terminal component: refer to Fig. E.6.3 for the assignment of variables. Determine the conductance parameters of the component, and write down the equations relating the two currents and two voltages.

8. For the 3-terminal component whose current-voltage relations are embodied in the APL function *COMV6*, explore the current response to all combinations of the values of *V*[1] and *V*[2] indicated by the vectors

```
V1←¯10 ¯5 0 5 10
V2←9 7 5 3
```

When *V*[1]←¯5 and *V*[2]←7, what is *I*[1] ? For what values of *V*[1] and *V*[2] is *I*[2]=105 ? What is the maximum recorded value of *I*[1] ? What is the minimum recorded value of *I*[2] ? What is the maximum recorded ratio of *I*[1] and *I*[2] ? What is the maximum recorded value of *V*[1]÷*I*[1] ? What is the minimum recorded value of *V*[1]÷*I*[2] ?

(8) 9. From the measurements carried out on the component *COMV6* in exercise 8, deduce the values of the conductance parameters provided you are satisfied that it is a linear and homogeneous component. Propose a model of the current-voltage relation of the component, incorporate it in an APL function, and test it.

10. A 3-terminal component is simulated by the APL function *COMV7*. By means of simulated measurements, determine a satisfactory linear homogeneous model relating the currents and voltages. Incorporate your model in an APL function, and carry out tests to see over what range of *V*[1] and *V*[2] the current entering terminal 1 is predicted to within an accuracy of 0.05 amp. If you have difficulty in doing this, try entering the expression

```
' *'[1+0.05<|(MODEL AV)-COMV7 AV]
```

where *MODEL* is your model of the component, and *AV* is the array of voltage excitation (a 3-dimensional array) employed in your simulated measurements.

11. A 3-terminal component is simulated by the APL function *COMV7* whose argument may only be a 2-element vector (i.e., other arrays whose first dimension is 2 will be rejected). With the knowledge that the function is linear and homogeneous, carry out the smallest number of measurements that is necessary to enable you to determine the conductance parameters. Check that the parameter values you deduce are correct.

12. By means of exploratory measurements on the component simulated by the function *COMV9* (which will only accept a 2-element vector as its argument), determine the voltage response when *I*[1]←8 and *I*[2]←¯2.

13. Measurements on a 3-terminal component show that a linear and homogeneous relation exists between its currents and voltages, and that this relation is described by the conductance matrix whose elements are: *G*[1;1]←0.05, *G*[1;2]←¯0.01, *G*[2;1]←4, *G*[2;2]←0.1. The component is now subjected to the current excitation shown in Fig.6.8, with *I*[1]←0.12 and *I*[2]←0.17. What are the values of

V[1] and *V*[2] ?

14. Two voltage excitations were applied to a linear and homogeneous 3-terminal component, and two current responses observed, as recorded below: determine the conductance matrix describing the component.

```
      COMPONENT 2 1
10  1
      COMPONENT ¯1 4
¯14 31
```

15. The conductance matrix describing a linear and homogeneous 3-terminal component is

```
6   3
5   1
```

What voltage excitation is required to ensure that 1 ampere enters terminal 1 and that the same current leaves terminal 3 ?

16. For a linear homogeneous component whose observed currents and voltages are as shown in Fig. E.6.3, the conductance matrix is

```
 4   ¯1
10    3
```

Determine the new conductance matrix describing the relation between the currents and voltages shown in Fig. E.6.5, for the same component. Test your answer.

17. Obtain a plot of *I*[1] versus *V*[1] and of *I*[2] versus *V*[2] for the 3-terminal components whose current-voltage relations are simulated by the APL functions *COMV*10, *COMV*11, *COMV*12 and *COMV*13. Discuss briefly how the value of a particular conductance parameter affects the appearance of the plots. Determine the resistance matrix for each of the components, and see if plots of voltage response versus current excitation exhibit any characteristic features.

18. The PNP transistor whose symbol is shown in Fig. E.6.6 is modelled by the APL function *PNPV* which accepts a voltage array with first dimension equal to 2 as its single argument, and returns the current response. By means of simulated measurements, obtain a model of the transistor. Test your model by incorporating it within an APL function.

19. A 3-terminal component is modelled by the function *COMV*14. Determine whether the component is active or passive.

20. Define a dyadic APL function which tests a 3-terminal component for activity. The left-hand argument is the

3-dimensional array of excitation for which the supply of power is to be investigated, and the right-hand argument is the character string which is the name of the function simulating the 3-terminal component.

21. A circuit having seven terminals is known to exhibit a linear and homogeneous relation between the currents entering terminals 1 to 6 and the voltages of these terminals with respect to the seventh terminal. This relation is embodied in the locked function *CIRCUITV*. By means of a single execution of this function, determine the conductance matrix which completely describes the circuit.

22. Fig. E.6.7 is a circuit containing two conductances and two voltage-controlled current sources. The latter are similar to ideal current sources except that their value, instead of being independent, is a linear and homogeneous function of a voltage. Apply Kirchhoff's current law at terminal 1 and again at terminal 2. Show that the circuit is described by a relation of the form `I←G+.×V` where there is a one-to-one correspondence between the elements of *G* and the parameters of the circuit. In other words, show that the circuit of Fig. E.6.7 can be used to model any linear and homogeneous 3-terminal component.

(22) 23. For the linear homogeneous 3-terminal components simulated by the functions *COMV5*, *COMV6*, *COMV11* and *COMV12* write down a model containing voltage-controlled current sources and conductances, and assign values.

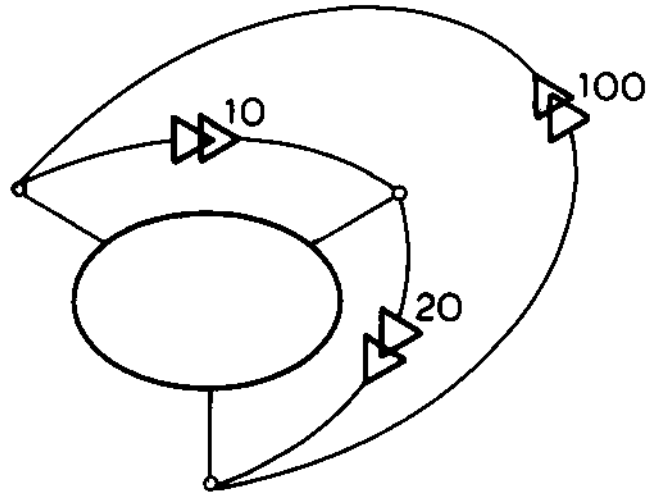


Fig. E. 6·1

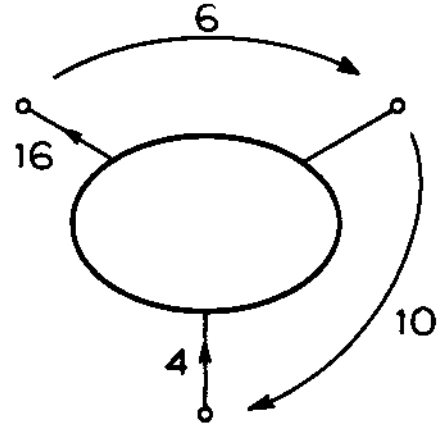


Fig. E. 6·2

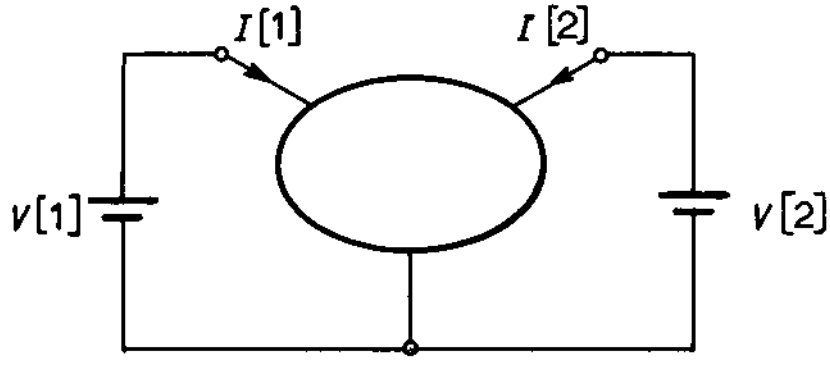
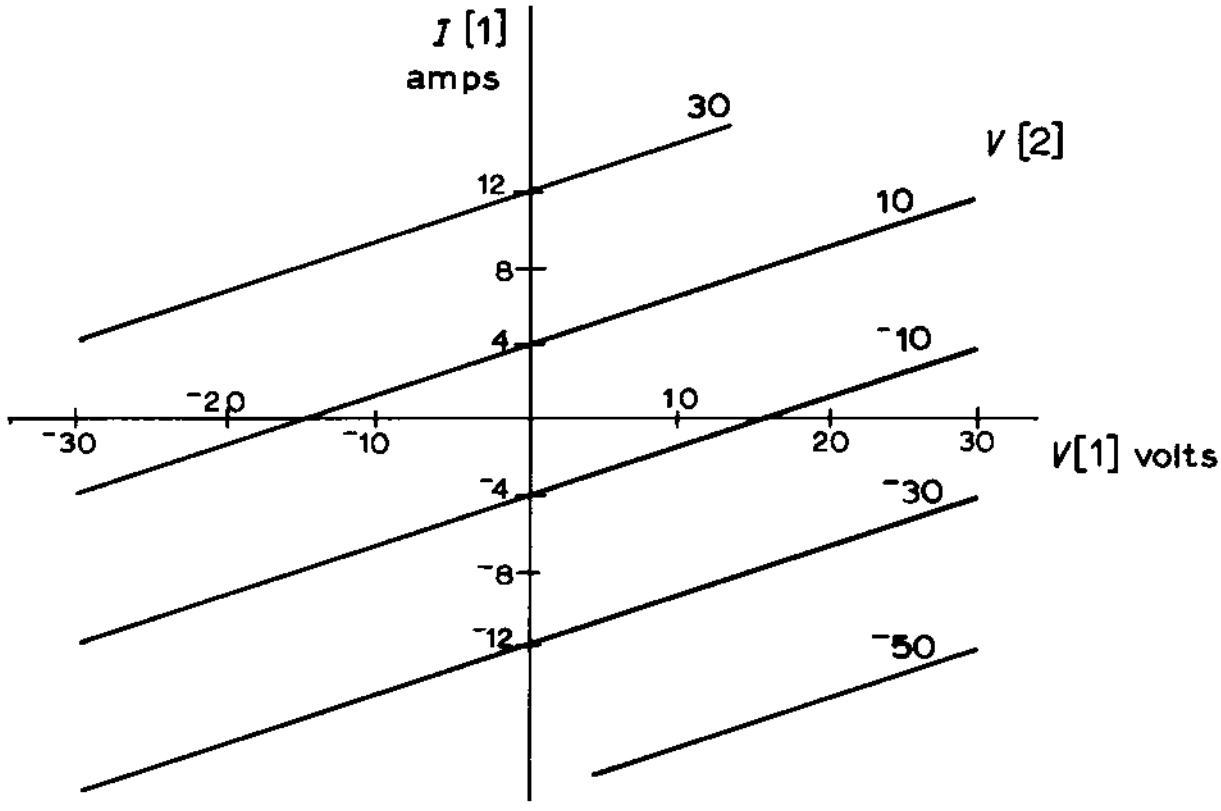


Fig. E. 6·3


I [1]
amps
30
V [2]
12
10
8
4
-10
-20
10
-30
-10
20
30
V[1] volts
-4
-30
-8
-12
-50

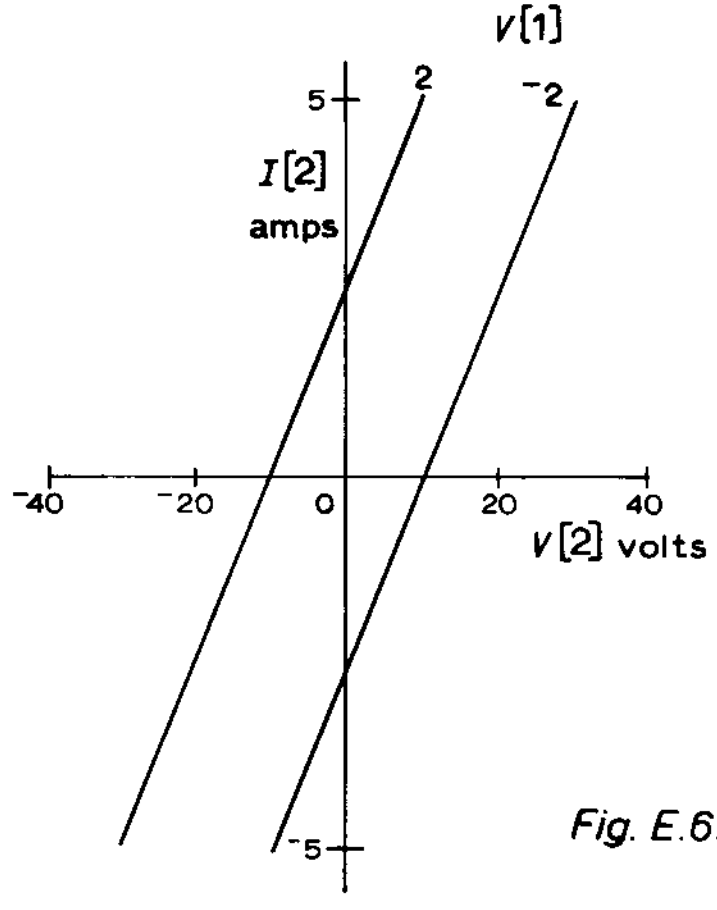

V[1]
2
-2
5
I[2]
amps
-40
-20
0
20
40
V[2] volts
-5

Fig. E.6.4

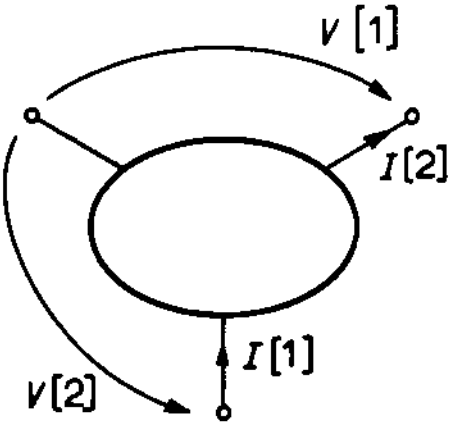


Fig. E.6·5

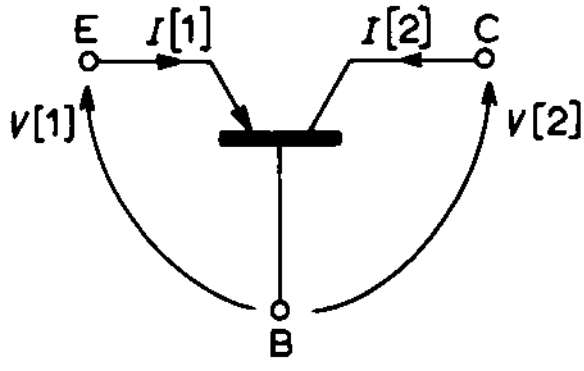


Fig. E.6·6

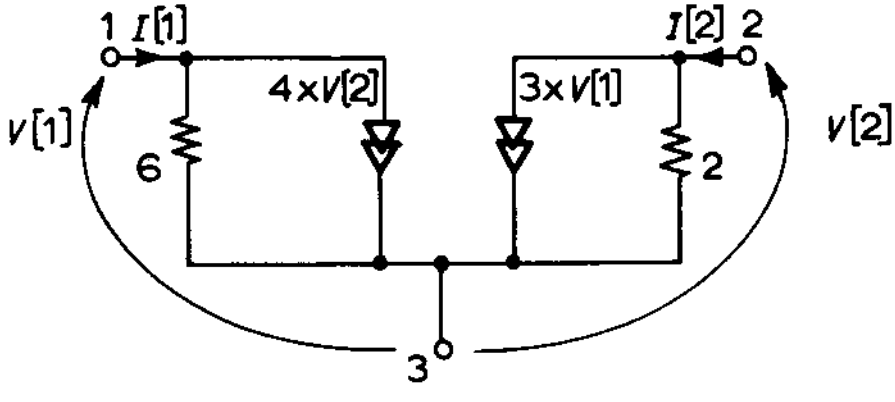


Fig. E.6·7

EXERCISES: CHAPTER 7

1. Draw a circuit composed of 2-terminal components. Draw its graph, and assign reference directions to the branches. Write down the branch connection matrix describing the directed graph.

2. Draw all the possible different branch-node representations of a 3-terminal component. Also draw two of the many possible branch-node representations of a 4-terminal component. In each case check that knowledge of the branch currents and voltages so identified is sufficient to determine <u>all</u> the terminal currents and terminal-pair voltages.

3. A directed graph can be described by its branch connection matrix. Propose two other methods of describing a directed graph, and illustrate them with examples.

4. The following are three branch connection matrices:

```
1 4 4 2 5 1      2 2 4 1 4 1      1 3 3 2 1
2 6 5 3 6 3      3 1 3 3 2 4      3 3 2 1 2
```

   Draw the corresponding directed graphs.

5. Fig. E.7.1 shows 3 directed graphs. Assign node and branch numbers and write down the branch connection matrix for each graph. Obtain the incidence matrix of each graph.

6. Fig. E.7.2 shows three circuits. For each circuit, write down a directed graph, assign node and branch numbers, and hence obtain an incidence matrix.

7. Define an APL function which is the inverse of *INCID*. In other words, a function which takes an incidence matrix as its argument and returns the branch connection matrix. Are there any graphs for which your defined function will not act as the inverse of *INCID* ? Test your answer to this question, and your defined function, by means of the examples of exercises 4 and 5.

8. Devise a test by means of which the description of a graph can be examined to see if it consists of separate

parts (for example, as in Fig. E.7.1(c)).

9. Draw a directed graph containing at least four nodes, and determine two sets of branch currents which obey Kirchhoff's current law.

10. Define a dyadic function *TESTKCL* which tests to see if an incidence matrix and a vector of branch currents are compatible with Kirchhoff's current law, it being assumed that no external nodal excitation current sources are connected to the circuit.

11. Draw the diagram of a circuit containing at least five 2-terminal components and three connection points. Draw a directed graph of the circuit, and arbitrarily assign values to the branch currents (i.e., do not worry if they do not obey Kirchhoff's current law at each node). What excitation current sources must now be connected to the nodes to ensure that Kirchhoff's current law is obeyed ? Draw these sources on your diagram. Can one of these sources be dispensed with ?

12. You are handed a piece of paper on which is printed the following array:

```
 1  1  0  0  0  1 ¯1  0
 0  0  1  1  0  0  1  0
 0 ¯1  0 ¯1  0  0  0 ¯1
¯1  0 ¯1  0  1  0  0  0
```

You are told that it was intended to print the whole of the incidence matrix of a 5-node, 8-branch directed graph, but that the last row was omitted by mistake. If possible, find the missing last line and hence draw the directed graph.

(12) 13. It was found in exercise 12 that an incidence matrix from which a row has been omitted, and which is called a reduced incidence matrix, *AR*, is a sufficient description of a directed graph. Under the assumption that it is the last row of the incidence matrix that has been omitted to obtain *AR*, express Kirchhoff's laws in terms of *AR* for a circuit with nodal current excitation. For a circuit with N nodes and B branches, what are the dimensions of the vectors of branch and nodal voltage and current involved in your new statements of Kirchhoff's laws ? Test these statements by means of a numerical example.

14. Can any row of an incidence matrix be removed and still allow the remainder to carry all the information about the directed graph ?

(3) 15. In exercise 3 you proposed alternative methods of describing a directed graph. Express both Kirchhoff's current law and Kirchhoff's voltage law in terms of these descriptions.

16. For the circuit shown in Fig. E.7.2(a) draw a directed graph and determine the incidence matrix. Arbitrarily assign values to the nodal potentials. Now determine, in a systematic manner, the values of the branch voltages. Repeat the calculation with each node potential raised by 10 volts. Now assume that each component is linear and homogeneous, and arbitrarily assign conductance values. From knowledge of the branch voltages, determine the branch currents. Systematically calculate the currents that must be injected at the nodes if Kirchhoff's current law is to be satisfied.

17. For the circuit of Fig. E.7.2(b) draw a directed graph. Arbitrarily choose a vector*IB*1 of branch currents and, by means of the *INCID* function if you wish, determine the vector *I*2 of nodal excitation currents which must then exist if Kirchhoff's current law is to be obeyed. Now choose a different vector *IB*2 of branch currents, and compute the corresponding vector *I*2 of injected nodal currents. What relation exists between $\Delta IB \leftarrow IB1 - IB2$ and $\Delta I \leftarrow I1 - I2$? Repeat the investigation to verify that changes in nodal and branch voltages also obey Kirchhoff's voltage law.

18. Define a function which accepts the description of a directed graph as its single argument, and returns the node pairs between which no branch is connected.

19. Consider a directed graph representing a circuit. For given values of nodal injected currents is there, in general, one or more sets of branch currents which, together with the injected currents, obey Kirchhoff's current law ?

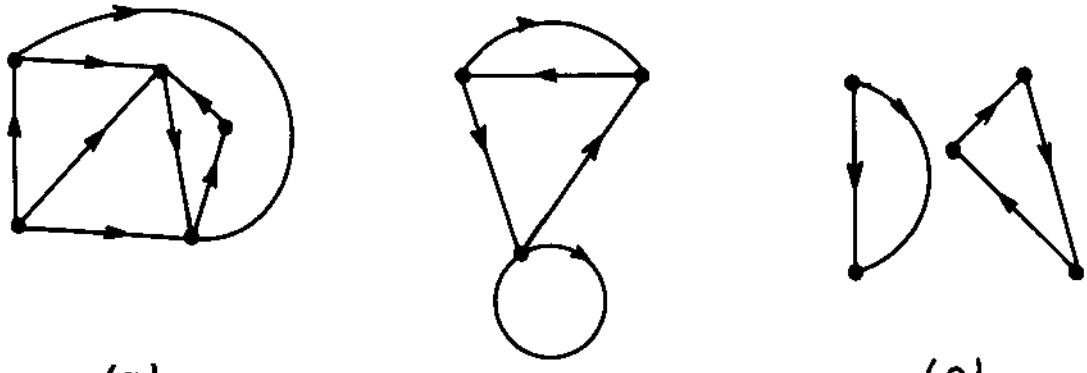


Fig. E.7.1

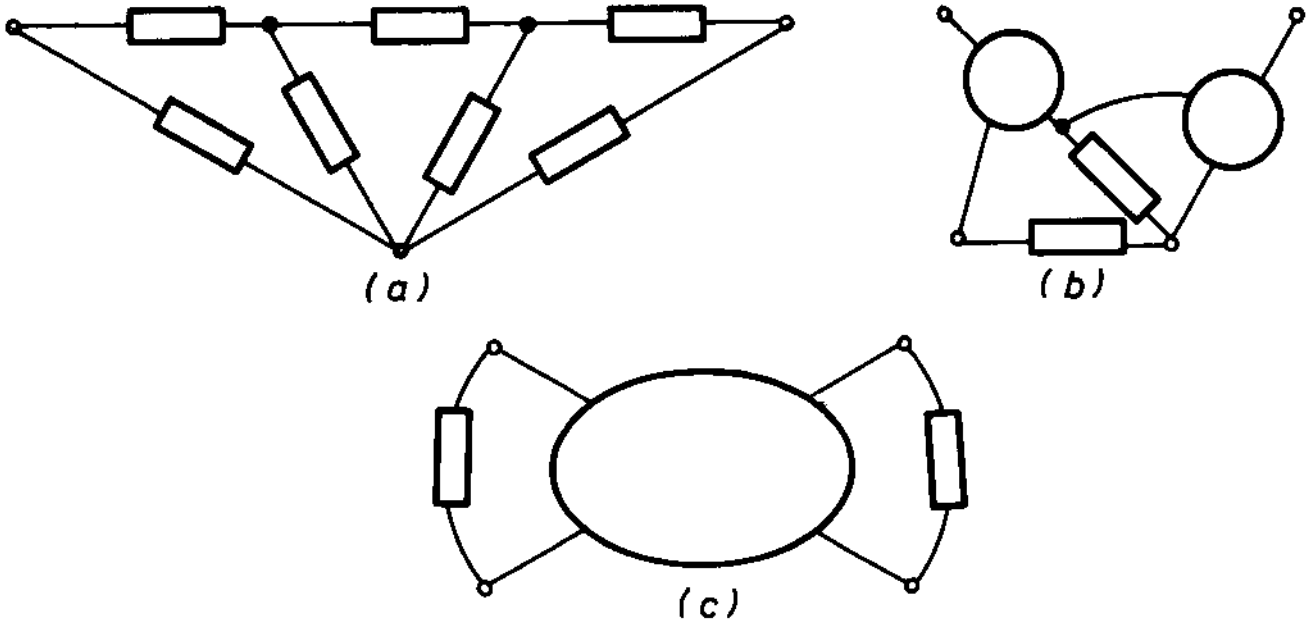


Fig. E.7.2

EXERCISES: CHAPTER 8

1. Four linear homogeneous components, together with their conductance values, are shown in Fig. E.8.1. Number the components, and write down the branch conductance matrix describing this collection of components.

2. For the set of components shown in Fig. E.8.1 choose an arbitrary vector of branch voltages and compute the associated vector of branch currents.

3. Use the function *COND* to generate the branch conductance matrix of the set of components shown in Fig. E.8.1. Define an arbitrary vector of branch voltages and hence compute the vector of branch currents.

4. For the collection of components shown in Fig. E.8.2 draw the corresponding directed graphs, number the nodes and branches, and write down the branch conductance matrix. Was it necessary to number the nodes ?

5. The branch conductance matrix describing a collection of linear homogeneous components is

```
8 0 0 0
0 2 0 0
0 0 7 3
0 0 4 9
```

Draw the directed graphs associated with the components, and then the component symbols with reference points and conductance descriptions.

(5) 6. Arbitrarily choose a 4-element vector of branch voltages associated with the components appearing in your answer to exercise 5. Carry out a calculation which enables you to show on your diagram (containing component symbols) the current entering every terminal.

7. If *YB* is the branch conductance matrix describing a collection of linear homogeneous components, does ⍴*YB* carry any information concerning the number of components in the collection ?

8. Use the function *AA* to generate the branch conductance matrix associated with the components in Fig. E.8.2.

9. In exercise 7.13 it was discovered that Kirchhoff's laws could be expressed in terms of the reduced incidence matrix *AR* as

```
IR←AR+.×IB
VB←VR+.×AR
```

where *AR* is normally obtained by deleting the last row of the incidence matrix, and *IR* and *VR* are called the reduced vectors of nodal current and voltage. Combine the two Kirchhoff laws so expressed with the branch relations *IB←YB+.×VB* to obtain relations, similar to equations 8.10 and 8.11, but expressing the reduced nodal conductance matrix *YR* as a function of *AR* and *YB*. Test the relation you propose by using the example of Fig.8.6.

10. For each of the circuits shown in Fig. E.8.3 obtain the incidence matrix and branch conductance matrix and, by matrix multiplication, the complete nodal conductance matrix.

(10) 11. Using the results of exercise 10 determine, by the appropriate matrix calculations, the currents labelled *IA*, *IB*, *IC* and *ID*.

12. In exercise 7.3 you proposed alternative means of describing a directed graph. See if you can readily discover expressions for the complete nodal conductance matrix of a circuit involving these alternative descriptions together with some description of the constituent components.

13. Obtain the complete nodal conductance matrix of the circuit shown in Fig. E.8.4(a). The entire circuit is now represented by the component symbol shown in Fig. E.8.4(b). For the reference terminal and node ordering implied by the dot and arrow, determine the reduced nodal conductance matrix.

14. If the complete nodal conductance matrix is determined from the relation *Y←A+.×YB+.×⍉A*, will a different result be obtained if the 1 and ¯1 entries in *A* are interchanged ? Test your answer by means of a numerical example.

15. For the circuit shown in Fig. E.8.5(a) define an incidence matrix and a branch conductance matrix, and hence obtain the complete nodal conductance matrix. What current would flow into terminal 2 if the voltage excitation shown in Fig. E.8.5(b) were applied ?

16. In the course of developing a computer program for circuit analysis an engineer wishes to see if the program is correctly computing the complete nodal conductance matrix. For one test circuit this matrix is computed to be

```
 21  ¯19  ¯2
  6    8  ¯14
¯27   11   17
```

Comment on this result.

17. With each node of the circuit shown in Fig. E.8.6 is associated a nodal voltage and an excitation current. For each node in turn express Kirchhoff's current law in terms of the excitation current, the nodal voltages and the conductances, whose values are shown. By examining your result, deduce a rule for writing down, by inspection, the complete nodal conductance matrix of a circuit containing only 2-terminal linear homogeneous components.

18. A 3-terminal component is described by the conductance matrix

```
10   ¯5
¯1    7
```

if terminal 3 is chosen as the reference terminal and the terminals are ordered clockwise. Determine the 2 by 2 conductance matrix describing the same component if terminal 1 is chosen to be the reference terminal, with the terminals still ordered clockwise. Can a complete nodal conductance matrix be associated with the component ? If so, find it, and say how it is related to the conductance matrices appropriate to different reference terminals.

(6.22)

19. A device called an operational amplifier, whose symbol is shown in Fig. E.8.7(a) can be modelled by the circuit shown in Fig. E.8.7(b). Define a 2 by 2 matrix *OPAMP* which is the conductance matrix of this model.

(19) 20. By means of the functions *AA*, *INCID* and *ZEDR* obtain the reduced nodal conductance matrix of the circuit of Fig. E.8.8, where each operational amplifier can be modelled by the circuit of Fig. E.8.7(b).

21. In exercise 9 it was discovered that the reduced nodal conductance matrix is related to the branch conductance matrix and the reduced incidence matrix by the expression *YR←AR+.×YB+.×⍉AR*. Find an expression similar in form to the above, but involving two different reduced incidence matrices obtained by deleting different rows from the incidence matrix, the result being a reduced nodal conductance matrix which would be obtained if the row and column deleted from the complete nodal conductance matrix were associated with different nodes. By means of a numerical example, show what currents and voltages are related by such a reduced nodal conductance matrix.

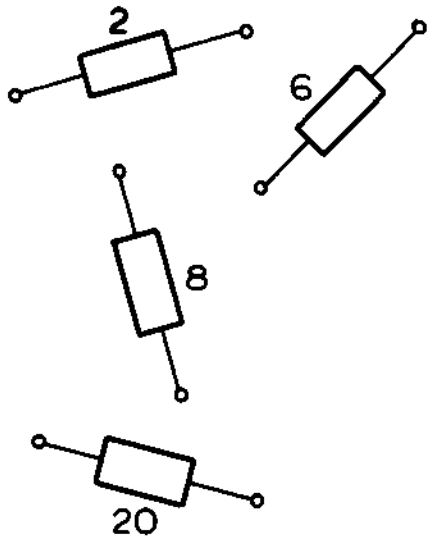


Fig. E. 8·1

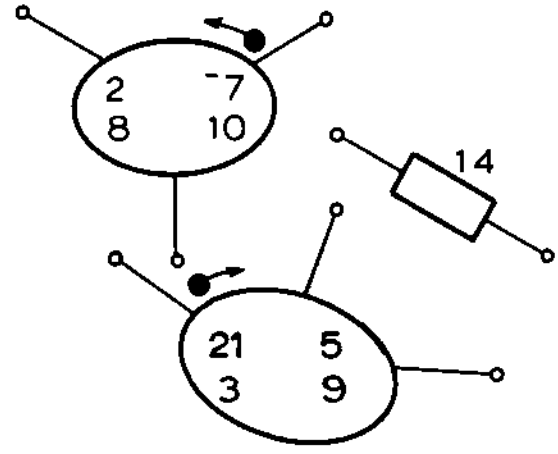
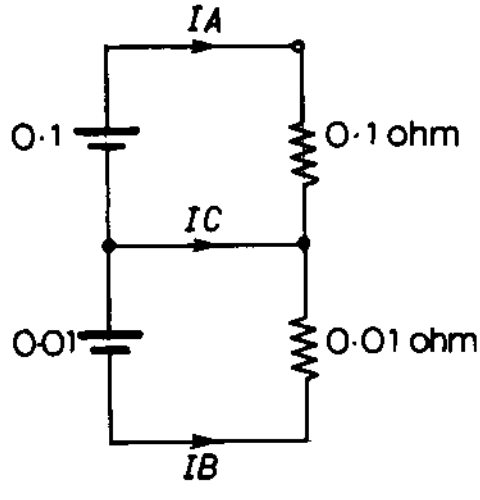


Fig. E. 8·2



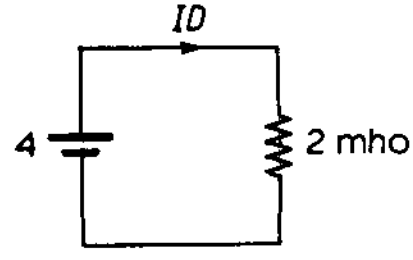
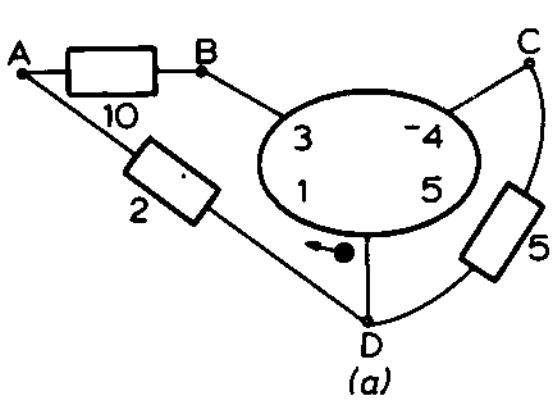
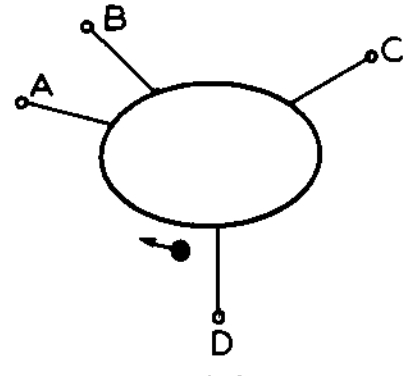


Fig. E. 8·3

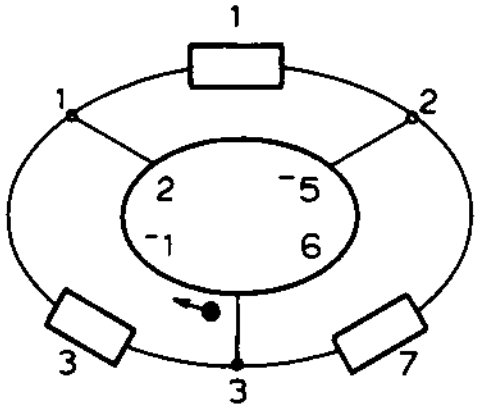


(a)

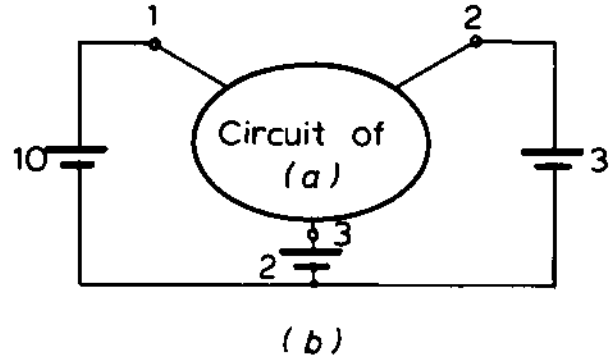


(b)

Fig. E. 8·4



(a)



(b)

Fig E. 8. 5

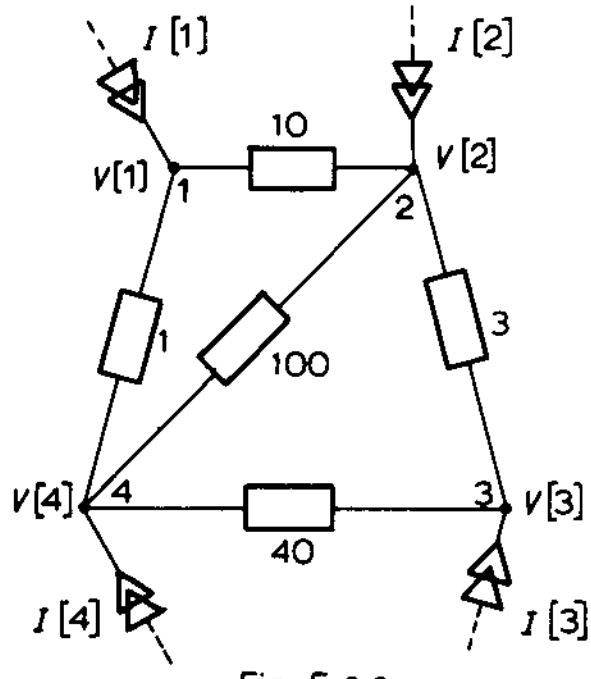
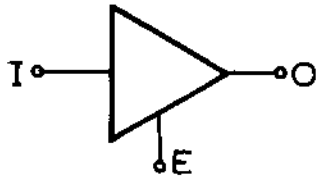
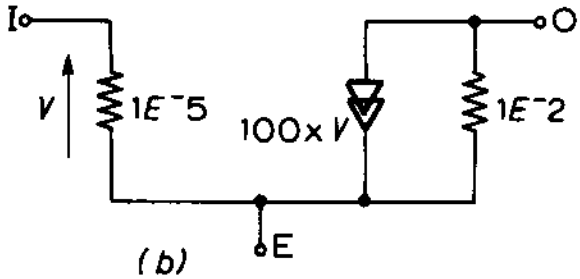


Fig. E. 8·6



(a)



(b)

Fig. E. 8·7

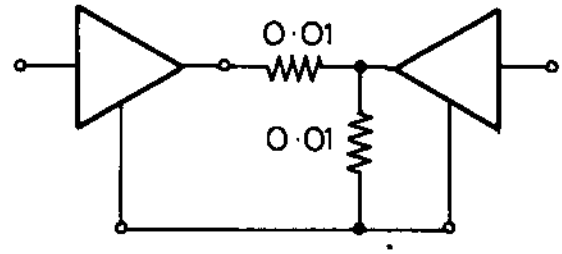


Fig. E. 8·8

EXERCISES: CHAPTER 9

1. By the simple application of Ohm's law and Kirchhoff's laws determine the currents flowing in each of the linear homogeneous components shown in Fig. E.9.1, as well as the currents entering terminals 1, 2 and 3 via the voltage sources.

2. The circuit of Fig. E.9.1 contains linear homogeneous 2-terminal components whose conductance values are shown. Obtain a branch conductance matrix and an incidence matrix for the circuit, and hence the complete nodal conductance matrix (use the functions *COND* and *INCID* if you wish). Determine the currents flowing into terminals 1, 2 and 3 via the voltage sources. Then determine, by the systematic application of Kirchhoff's laws and Ohm's law, the branch currents. What change will occur in the nodal and branch currents if (a) each voltage source is increased in value by 100 volts ? (b) each voltage source has its value doubled ? (c) each conductance has its value multiplied by ten ? Test your answers.

3. Draw the circuit of Fig. E.9.1 <u>without</u> excitation. Obtain its reduced nodal conductance matrix by considering it to be a 3-terminal component, selecting terminal 3 as the reference point, and directly calculating the conductance parameters of this component (e.g., to find $G[2;1]$ which is defined as $I[2]\div V[1]$ when $V[2]\leftarrow 0$, consider a short-circuit to be connected between terminals 2 and 3 and a one-volt source between terminals 1 and 3, and determine $I[2]$ and $V[1]$ by the simple application of Kirchhoff's laws and Ohm's law). If exercise 2 was attempted, compare answers.

4. The circuit of Fig. E.9.1 is subjected to the nodal current excitation shown in Fig. E.9.2. Compute the values of $V[1]$ and $V[2]$, and hence the component voltages and currents. Check that Kirchhoff's laws are obeyed. What power is dissipated in each component ? What is the total power dissipated by the components ? Should this be equal to the power supplied by the current sources ? Test your answer. What change will occur in the response voltages if all conductance values are multiplied by ten ? Check your answer by means of a calculation.

5. Use two different methods to compute the value of the current *I* in the circuit of Fig. E.9.3, first replacing the circuits A and B by their Norton equivalents. First, express Kirchhoff's current law at each node in turn in terms of the nodal voltages, and then solve to find the nodal voltages. Second, use a systematic approach employing the functions *COND* and *INCID*.

6. Consider the circuit of Fig. E.9.4, which is shown in the following exercise to have an important practical application. Recall the expressions for the equivalent resistance of two resistances connected in series or parallel, and hence successively calculate the total resistance to the left of AA', BB', CC', DD' and EE'. If the terminal X of the one ampere current source is connected to terminal 3, what is the value of *V* ? If, instead, the source is connected to terminal 2, what is the value of *V* ? Repeat for the source connected to terminal 1. What pattern can you identify in the successive values of *V* ?

7. The circuit of Fig. E.9.5 contains only linear homogeneous 2-terminal components, whose resistance values are shown. Its function is to act as a digital-to-analog (D/A) converter, by providing a voltage *V* whose value is proportional to a number represented in binary form by the state of the switches. If the left- and right-hand positions of the switches are represented by 0 and 1 respectively, the binary number represented by the switches in Fig. E.9.5(a) is 101 (5 to base 10). Thus, if switch S2 is now placed in the right-hand position, changing the binary number represented to 111 (7 to base 10), then *V* increases to (7÷5) times its former value. The circuit essentially performs the function of the weighted reduction of a vector to yield a scalar. Represent each resistance of 10 kilohms in series with a voltage of either 10 or 0 volts by the Norton equivalent of Fig. E. 9.5(b), where the value of *ISC* is one or zero milliamps respectively. By means of the functions *COND*, *INCID* and *ZEDR* obtain the reduced nodal resistance matrix *ZR* appropriate to the choice of terminal 4 as the reference terminal. Examine *ZR* to see if the circuit performs the stated function, and carry out a calculation or two to verify your conclusion.

8. Draw a directed graph of the circuit of Fig. E.9.6, and hence write down the appropriate incidence and branch conductance matrices. By matrix multiplication obtain the complete nodal conductance matrix, and hence the reduced nodal conductance matrix if terminal 3 is chosen as the reference terminal. Obtain the nodal voltage response to the excitation shown, and hence the value of *V*.

9. By using the functions *AA*, *INCID* and *ZEDR*, determine the voltage response *V* of the circuit shown in Fig. E.9.6. Also calculate the branch voltages and currents, and check that Kirchhoff's laws are obeyed. If you have attempted exercise 8 compare results.

10. By using defined functions, find the value of the current *I* in the circuit of Fig. E.9.7.

(9) 11. This exercise is designed to demonstrate the need to take care in the choice of branch reference current directions when two or more branches represent a component whose conductance matrix is given. Repeat the systematic analysis of the circuit of Fig. E.9.6, but under the following conditions:
    (a) the branch-node representation of the 3-terminal component is as shown in Fig. E.9.8(a),
    (b) the reference direction of <u>one</u> of the two branches is reversed, as in Fig. E.9.8(b),
    (c) the reference directions of <u>both</u> branches are reversed, as in Fig. E.9.8(c).
    In this way, show that the conductance matrix description of the 3-terminal component enters the branch conductance matrix <u>without</u> modification only under certain conditions.

12. What is the implication, for the calculated values of the following, of an accidental interchange of the entries 1 and ¯1 in the incidence matrix of a circuit:
    (a) the branch conductance matrix,
    (b) the complete nodal conductance matrix, as given by the expression *Y*←*A*+.×*YB*+.×⍉*A*,
    (c) the reduced nodal resistance matrix, as given by the expression *ZR*←⌹¯1 ¯1↓*Y*,
    (d) the predicted nodal response voltages, given by the expression *VR*←*ZR*+.×*IR*,
    (e) the predicted branch voltages and currents ?
    Test your answers by means of a calculation using the circuit of Fig. E.9.6.

13. Fig. E.9.9 shows a circuit employing an <u>ideal</u> operational amplifier, i.e., the input current *II* is zero and the voltage gain *VO*÷*VI* is minus infinity. By the simple application of Kirchhoff's laws and Ohm's law, show that the voltage gain *VO*÷*VIN* is equal to -*R2*÷*R1*. If the value of *VO* is always finite, what conclusion can be drawn about the value of *VI* ? Can you see why terminal I might be called the 'virtual-earth point', where 'earth' refers to the reference terminal E ?

14. In the circuit of Fig. E.9.10(a) the operational amplifier can, provided its voltages and currents remain within specified bounds, be modelled by the circuit of Fig. E. 9.10(b). By systematic means, determine the value of the overall voltage gain *VO*÷*VIN* when excitation is applied between terminals IN and E. For what value of *G* (the parameter describing the voltage-controlled current source

in the model of Fig. E.9.10(b)) does the value of *VO÷VIN* depart from the 'ideal' value of *-R2÷R1* by more than two percent ? With *G* at its nominal value of 1000, how large or small can the resistance *RO* become before the same departure occurs in the voltage gain ?

15. A variable gain amplifier employing an operational amplifier is shown in Fig. E.9.11. If the operational amplifier is ideal in the sense described in exercise 13, show that the overall voltage gain *VO÷VIN* is *-÷X×R2÷R1*.

16. Compute the overall voltage gain *VO÷VIN* of the variable gain amplifier shown in Fig. E.9.11 when *R1←R2←R3←1000* ohms, and for *X* equal to 0.3 and 0.8. Assume that, within the normal range of operation, the operational amplifier can be modelled by the circuit of Fig. E.9.10(b).

17. Attempt to compute the nodal voltage response, and then the branch voltages and currents, of the current excited circuits shown in Fig. E.9.12. The circuit in (a) is distinguished by the fact that it contains a self-loop, whereas that in (b) has a graph which consists of two parts. Comment on any difficulties or special characteristics of the analyses you perform.

18. Draw a circuit composed of 2-terminal resistances and a single excitation current source. Find the current flowing in each resistance. Now select one resistance and replace it by a current source whose value is identical in magnitude and direction with the current previously flowing in the resistance. The circuit now contains two current sources. Compute the current flowing in each resistance. Comment on the two sets of resistance currents that have been calculated. Can you formulate what might be called a 'substitution theorem' on the basis of your results ? Extend your exploration to circuits containing voltage-controlled current sources.

19. Two 3-terminal circuits A and B are to be connected in cascade, as shown in Fig. E.9.13(a), to form a new 3-terminal circuit, shown in (b) in dashed lines. Define an APL function *CASCADE* which takes the 2 by 2 reduced nodal conductance matrices *YRA* and *YRB* of the constituent circuits as its arguments, and returns the 2 by 2 reduced nodal conductance matrix of the cascade connection. (Note: in proceeding from the conductance description of the new four-terminal circuit to that of the three-terminal circuit in dashed lines, realize that no excitation current will be applied to terminal 2 and that the voltage of this terminal is of no interest. That this is the opposite situation to the one in which the choice of a reference point (*V←0*, *I* of no interest) leads to the deletion of a row and column of *Y* to obtain *YR* may suggest a way of tackling the above problem). Test your defined function with a number of circuits.

20. Carry out an analysis of the circuit of Fig. E.9.6 by employing the following expression for the reduced nodal conductance matrix:

```
YR←AR1+.×YB+.×⍉AR2
```

where *AR1* and *AR2* are reduced incidence matrices in which the deleted rows refer, respectively, to the terminals separately chosen as the terminals for current and voltage reference.

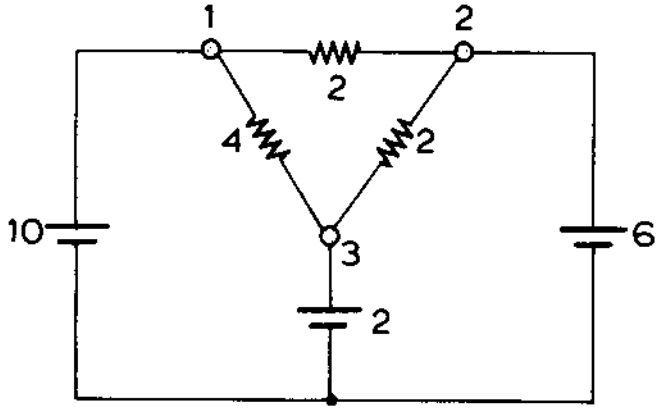


Fig. E. 9. 1.

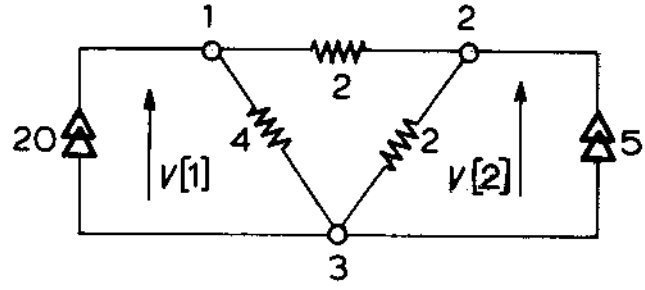


Fig. E. 9. 2.

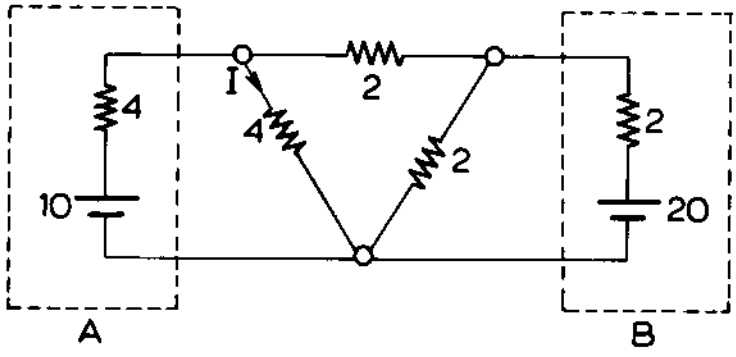


Fig. E. 9. 3.

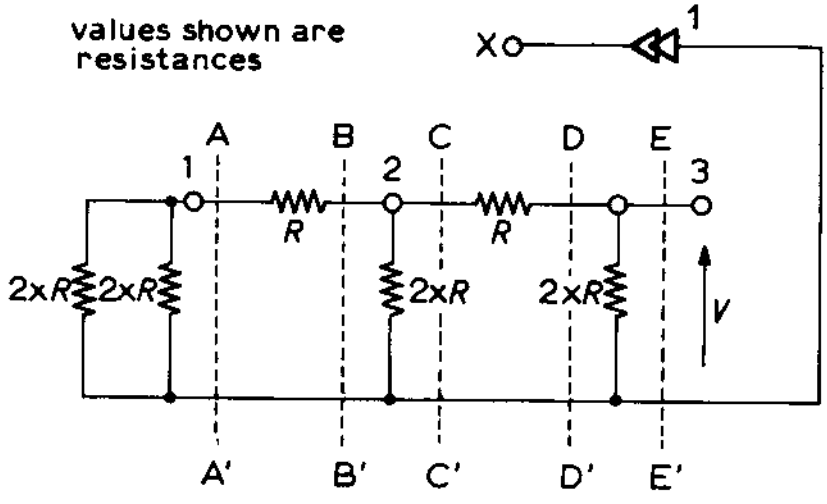
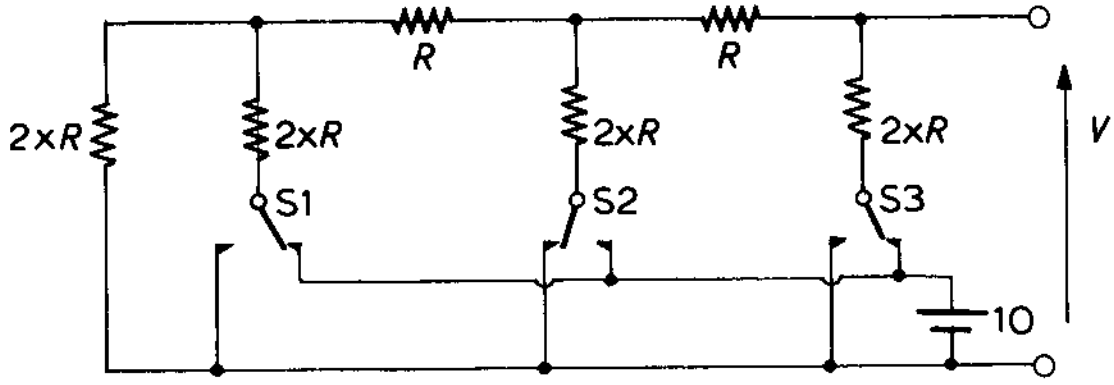
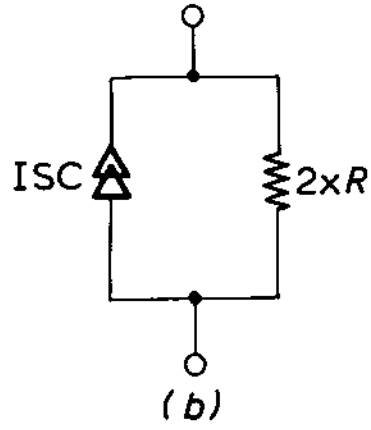


Fig. E. 9. 4.



(a)

R= 5 kilohms



(b)

Fig. E. 9. 5.

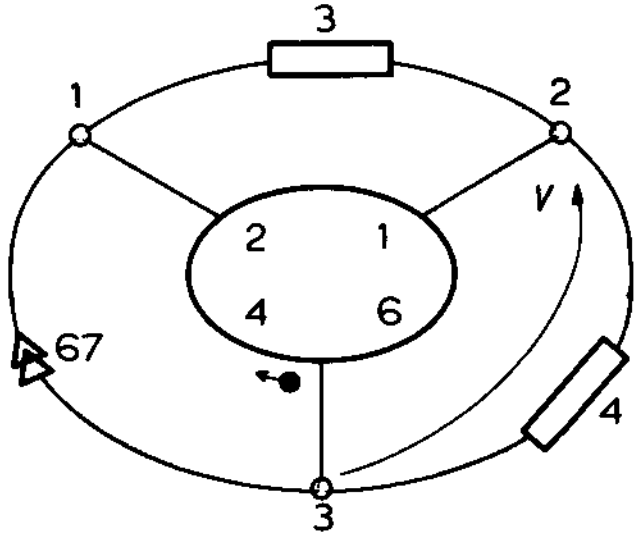


Fig. E. 9. 6.

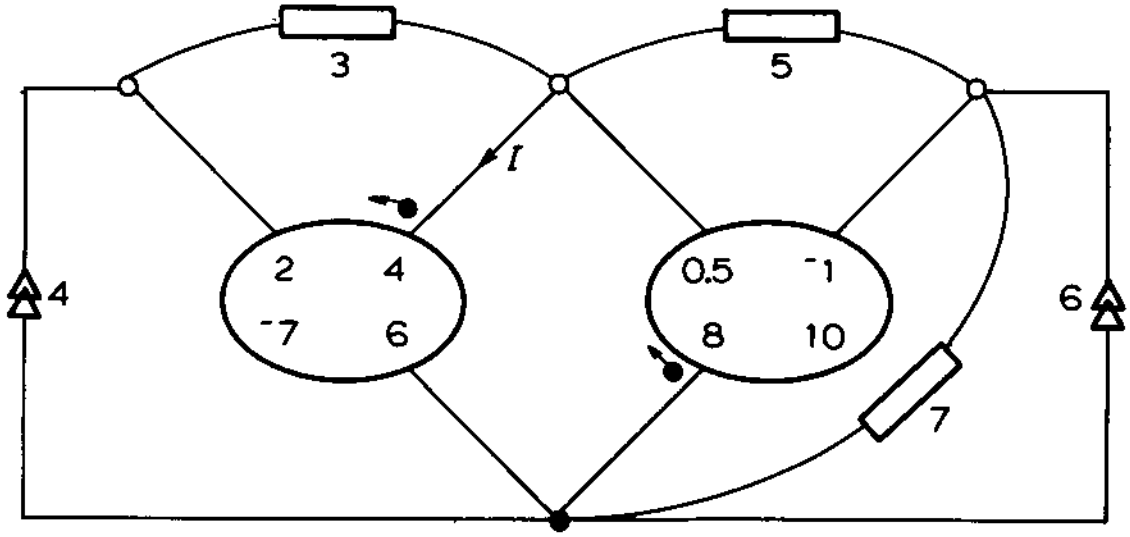


Fig. E. 9. 7.

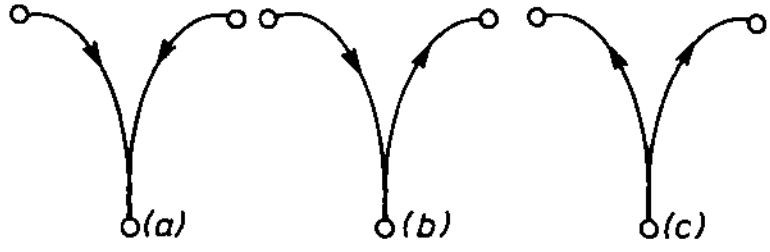
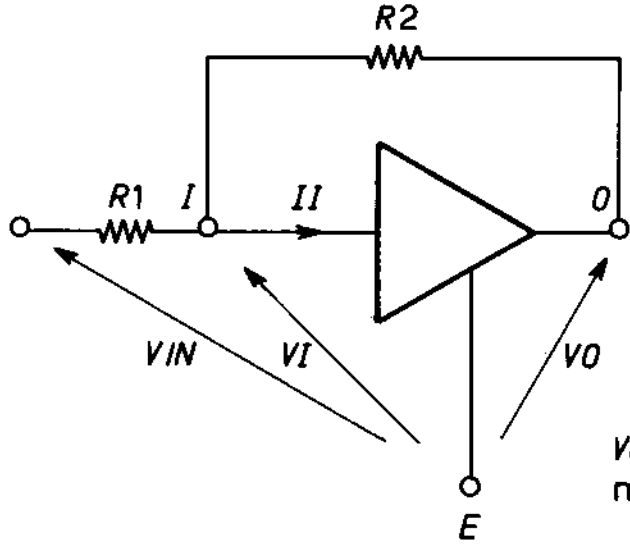


Fig. E. 9.8.



V0 ÷ VI has the valve minus infinity.

Fig. E. 9. 9.

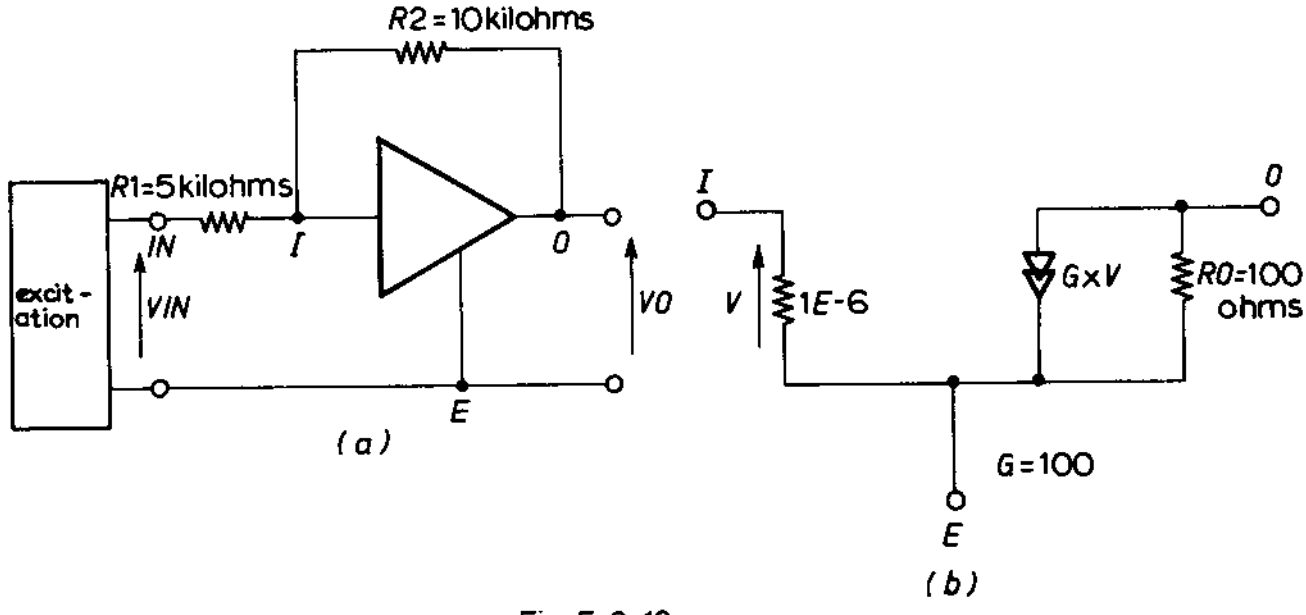


Fig. E. 9. 10.

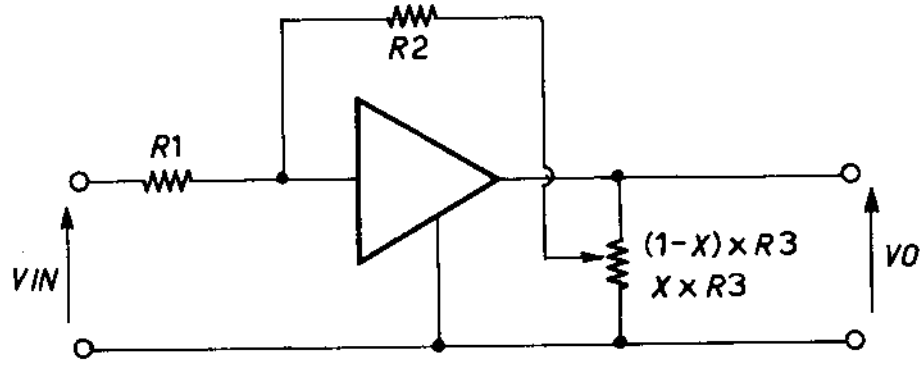


Fig. E. 9. 11.

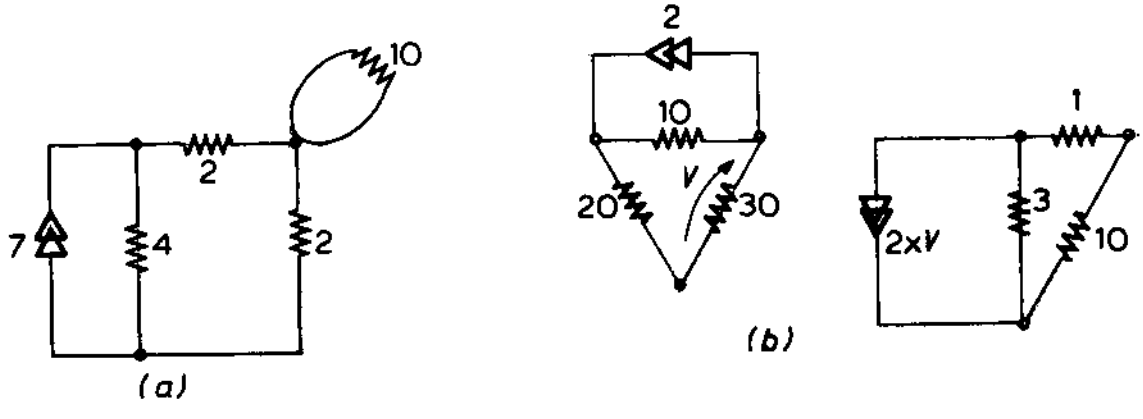
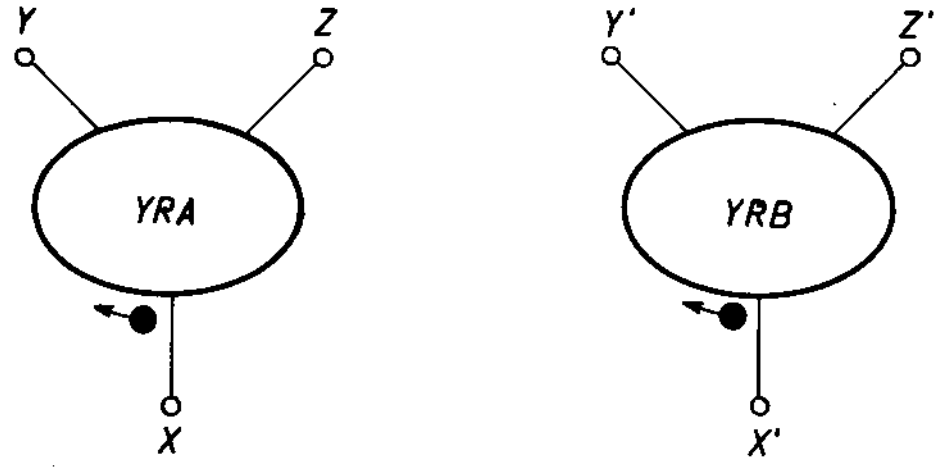
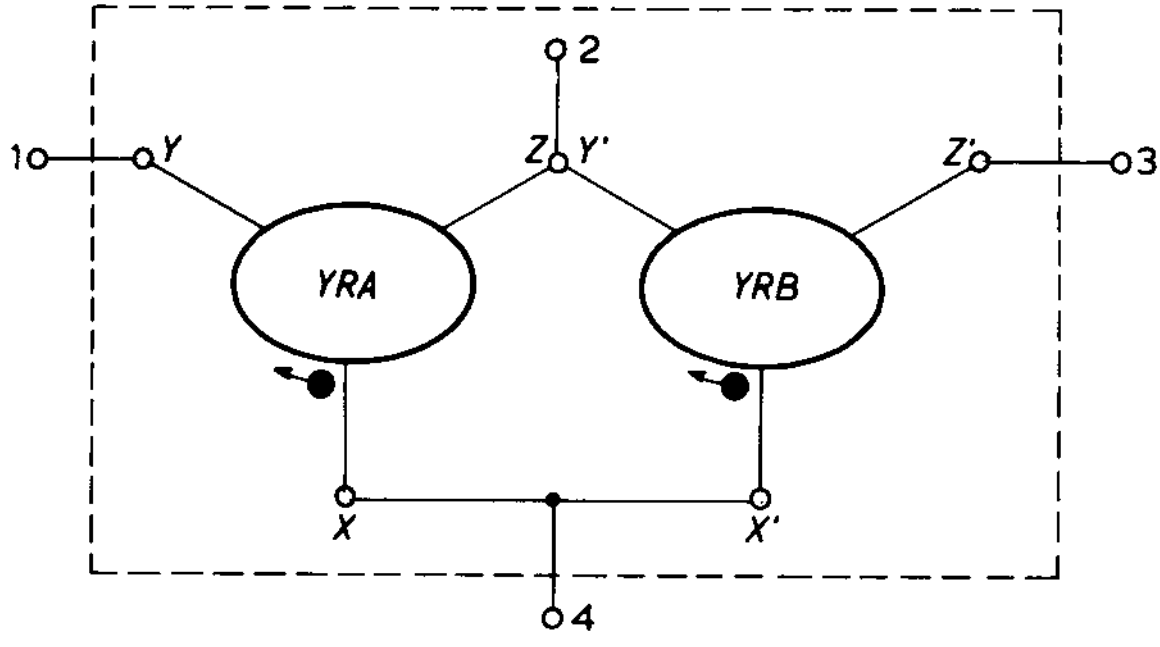


Fig. E. 9. 12.



Circuit A (a) Circuit B



(b)

Fig E. 9. 13.

EXERCISES: CHAPTER 10

1. Fig. E.10.1 shows a number of linear and homogeneous circuits: the conductance or resistance description of the circuit or its constituent components is provided in each case. Calculate the current, voltage and dissipated power where indicated by *I*, *V* and *P* respectively, and in each case state what element(s) of the complete circuit's reduced nodal resistance matrix is(are) essential to the calculation.

2. A linear homogeneous circuit possesses five terminals labelled A, B, C, D and E. When terminal E is selected as the reference terminal, the circuit is described by the following reduced nodal resistance matrix *ZR*:

$$\begin{matrix} ZAA & ZAB & ZAC & ZAD \\ ZBA & ZBB & ZBC & ZBD \\ ZCA & ZCB & ZCC & ZCD \\ ZDA & ZDB & ZDC & ZDD \end{matrix}$$

   State which element(s) of *ZR* must be known in order to calculate the voltage response *V* in each of the situations shown in Fig. E.10.2.

3. For the linear homogeneous circuit shown in Fig. E.10.3 calculate the following properties:

   (a) the voltage amplification *VOUT÷VIN*,
   (b) the input resistance *VIN÷IIN*,
   (c) the power gain, being the power supplied to *RL* divided by the power supplied by the excitation current source,
   (d) the current amplification *IOUT÷IIN*,
   (e) the total variation in input resistance (*VIN÷IIN*), as the conductance of *RL* varies between zero and infinity.

4. For the circuit of Fig. E.10.3, what is the maximum power available from the pair of terminals 2 and 3 if the excitation current has a value of 1 milliamp ? What value of resistance would have to be connected between these terminals in order to extract this power ?

(3) 5. The voltage amplification *VOUT÷VIN* of the circuit of Fig. E.10.3 was calculated in exercise 3. What voltage amplification would be obtained if the circuit were to be excited by an ideal voltage source rather than an ideal current source ?

6. Find the Thevenin equivalent circuit for the circuit of Fig. E.10.3 when terminals 2 and 3 are the accessible terminals, and the current excitation has the value of 1 milliamp. Also determine the parameters of the Norton equivalent circuit. If you attempted exercise 4, see if your answers are consistent.

7. A variable resistance *R* is connected between terminals 2 and 3 of the circuit of Fig. E.10.3, when the current excitation is 20 milliamps. Determine the current in the resistance as its value is varied over the values *R←10*16*. Plot the current versus *⊛R*.

8. When the current excitation of the circuit of Fig. E.10.3 is twenty amps , a nonlinear component having the current-voltage characteristic *I←V*3* is connected between terminals 2 and 3. What current will flow in this nonlinear component ?

9. Two measurements carried out on a linear homogeneous circuit are represented in Fig. E.10.4(a) and (b). What is the value of *V* when the circuit is excited as shown in Fig. E.10.4(c) ?

10. For a given linear and homogeneous circuit to which a number of current excitation sources are connected, it is found that a resistance connected between terminals A and B must assume a value of 1000 ohms if the power dissipated in the resistance is to be a maximum. Will the value of 1000 ohms necessarily remain the optimum value if the values of the excitation sources are changed, or if some are removed and others added ? Test your answers.

11. A certain linear and homogeneous circuit possesses two pairs of terminals. Associated with each pair of terminals is a 2-element vector of resistances *ROC*, *RSC* which are, respectively, the values of the input resistance at that pair of terminals when the other pair is connected to an open-circuit or short-circuit. By means of numerical examples investigate the relation between the values of *ROC* and *RSC* for the two terminal pairs.

12. Fig. E.10.5(a) is the model of a differential amplifier, the ideal version of which generates an output voltage which is the product of plus infinity and the difference between the two input voltages (in the sense indicated by the + and - signs), and whose input current is zero. One advantage of this device is that it ideally eliminates from the output all traces of any signal which is common to the two inputs, a signal which is often undesirable in some sense and which is called the common-mode signal. First, compute the reduced nodal resistance matrix of the circuit of Fig. E.10.5(b), and show that *V* is approximately

ten times the difference between *VX* and *VY* (both *R2÷R1* and *R4÷R3* are equal to 10). Now represent a common-mode signal by means of an identical voltage source *VCM* in series with *R1* and *R3* , as shown in Fig. E.10.5(c). By first obtaining a suitable Norton equivalent for these voltage sources in series with *R1* and *R3*, generate a reduced nodal resistance matrix which will demonstrate not only the constant amplification of the difference between *VX* and *VY*, but the fact that the output voltage is independent of *VCM*.

13. The purpose of this exercise is to introduce Tellegen's theorem. Draw two separate circuits bearing no other relation to each other than the fact that they have the same directed graph (i.e., the same topology) when both components and current excitation sources are regarded as branches. For the first circuit, with arbitrary values assigned to the conductances and excitation current sources, compute the vector *IB1* of branch currents (remembering that these will include one or more excitation currents). Now assign arbitrary values to the conductances and excitation currents of the second circuit, and for this circuit compute the vector *VB2* of branch voltages. Now compute the inner product

    ```
    +/IB1×VB2
    ```

    and comment upon its value. Show that the result you have obtained for a particular pair of circuits is generally valid by means of a proof similar to the proof of the energy conservation theorem in section 10.5.

14. In the interactive study of circuits, and in circuit design, it is often useful to be able to determine the parameters of the Thevenin equivalent circuit of a 2-terminal circuit. Devise a function, *THEVENIN*, which takes as its arguments the reduced vector *IR* of nodal excitation currents and the reduced nodal resistance matrix *ZR*, and which generates and prints the Thevenin parameters *VOC* and *RIN* under the assumption that the two accessible terminals are the (highest numbered) reference node and the next highest-numbered node. Test the function *THEVENIN* by means of an example. To make this function more useful (see exercise 18) it may be desirable to define two functions to re-order the elements of *IR* and the rows and columns of *ZR* to make any selected node, together with the reference node, the two terminals for which the Thevenin equivalent is to be found.

15. Repeat exercise 14 for the generation of a Norton equivalent circuit.

16. Investigate, by example, the validity of the energy conservation principle for a circuit containing at least one 3-terminal component.

17. For a linear homogeneous circuit, the superposition principle allows any voltage or current to be computed as the sum of components due to each excitation source considered in turn. By means of calculations on circuits, find a counterexample to the statement that the power dissipated in a resistor is the sum of the powers dissipated in the resistor due to each excitation source acting alone.

18. This question is concerned with the renumbering of nodes that is sometimes necessary in the course of circuit analysis (see exercise 14). Take a circuit having at least four nodes and obtain its reduced nodal resistance matrix *ZR* under the assumption that the highest numbered node is the reference node. Choose an arbitrary reduced vector *IR* of nodal current excitation, and compute the reduced vector *VR* of nodal voltages. Now define a new ordering of the node numbers by means of a vector *X*: thus

```
X←1  2  5  4  3
```

means that the numbering of nodes 3 and 5 is interchanged. From *ZR* and *IR* generate the new reduced nodal resistance matrix *ZRN* and the new reduced vector *IRN* of nodal excitation currents and, by computing the reduced vector of nodal voltages, check that the allocation of new node numbers has not led to an incorrect circuit analysis.

19. Fig. E.10.6 shows a circuit containing a differential amplifier. Under the assumption that the amplifier is ideal (i.e., *VO÷VI* is infinity and *II* is zero), determine the value of the resistance between terminals X and Y. What interesting property does this resistance possess ? Check your result by carrying out a calculation in which the differential amplifier is modelled by the circuit of Fig. E.10.5(a), and where *RF←R←*10 kilohm.

20. The very-low-frequency model of a field-effect transistor (FET) amplifier is shown in Fig. E.10.7(a). The FET itself is found to be capable of being modelled by the circuit of Fig. E.10.7(b) if *VDS*>1.5, *VGS*>¯1.5 and *ID*>1 milliamp. Assume that the input voltage *VS* is of square waveform and hence compute the signal voltage gain of the amplifier (i.e., the ratio of the time-varying components of the voltages *VS* and *VOUT*).

21. An ideal gyrator is a 3-terminal linear homogeneous component which can be characterized by a conductance matrix of the form

$$\begin{matrix} 0 & G \\ -G & 0 \end{matrix}$$

and is represented by the symbol shown in Fig. E.10.8(a).

Determine the conductance between terminals A and B in the circuit of Fig. E.10.8(b). What function does the gyrator perform ? You will have noticed that the reference dot and arrow were omitted from the symbol in Fig. E.10.8(a). Was this omission justified ? To answer this question, obtain, for the gyrator, the conductance matrix appropriate to the choice of a selection of reference terminals. In a practical gyrator it is not easy to achieve or maintain identical values for $G[1;2]$ and $G[2;1]$: if their nominal value is 0.5, what percentage difference can be tolerated if the conductance between terminals A and B is to be kept to within one percent of its nominal value ? Test your answer by means of a numerical example.

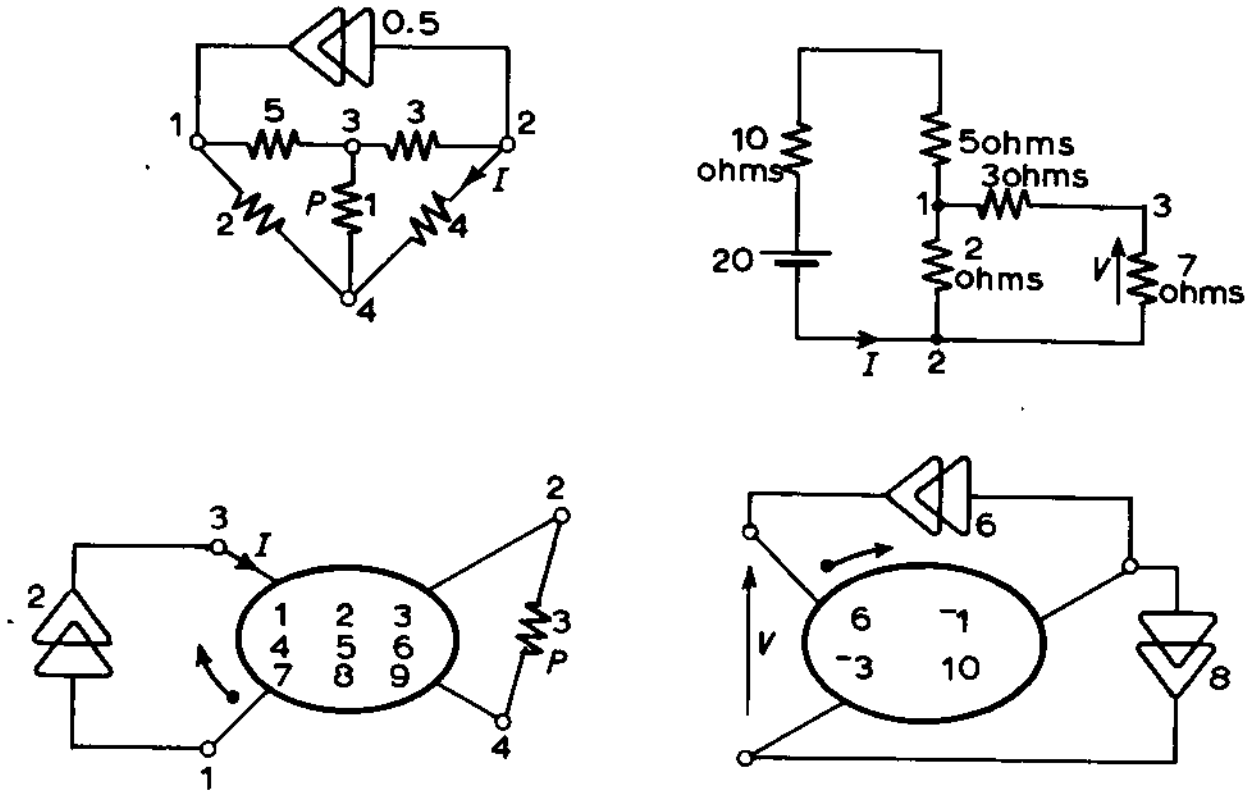


Fig. E. 10.1

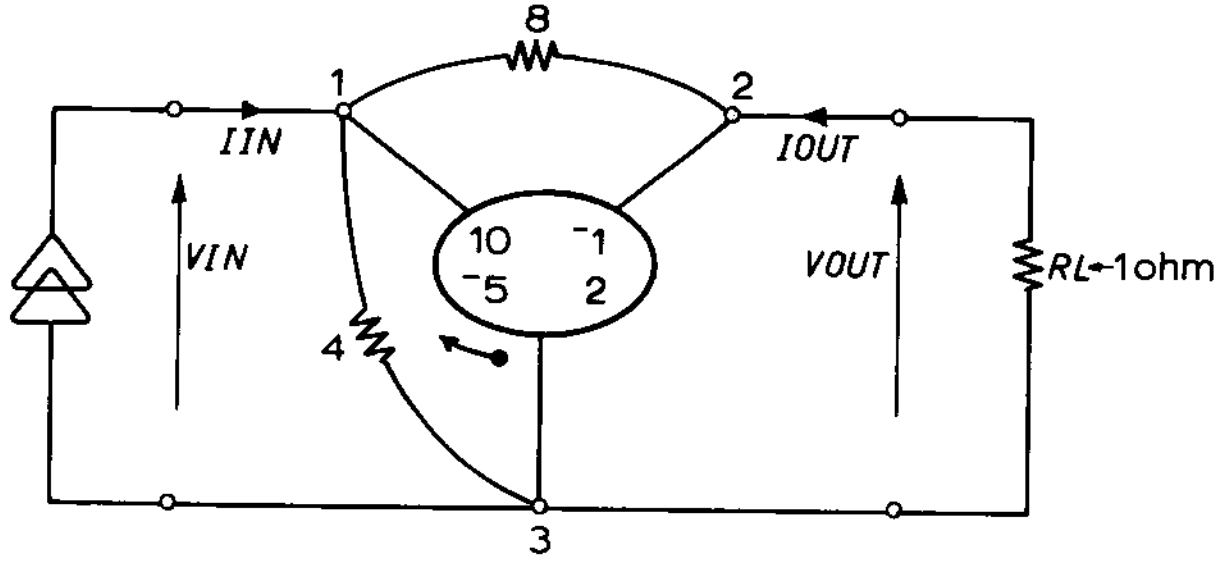


Fig. E.10.3

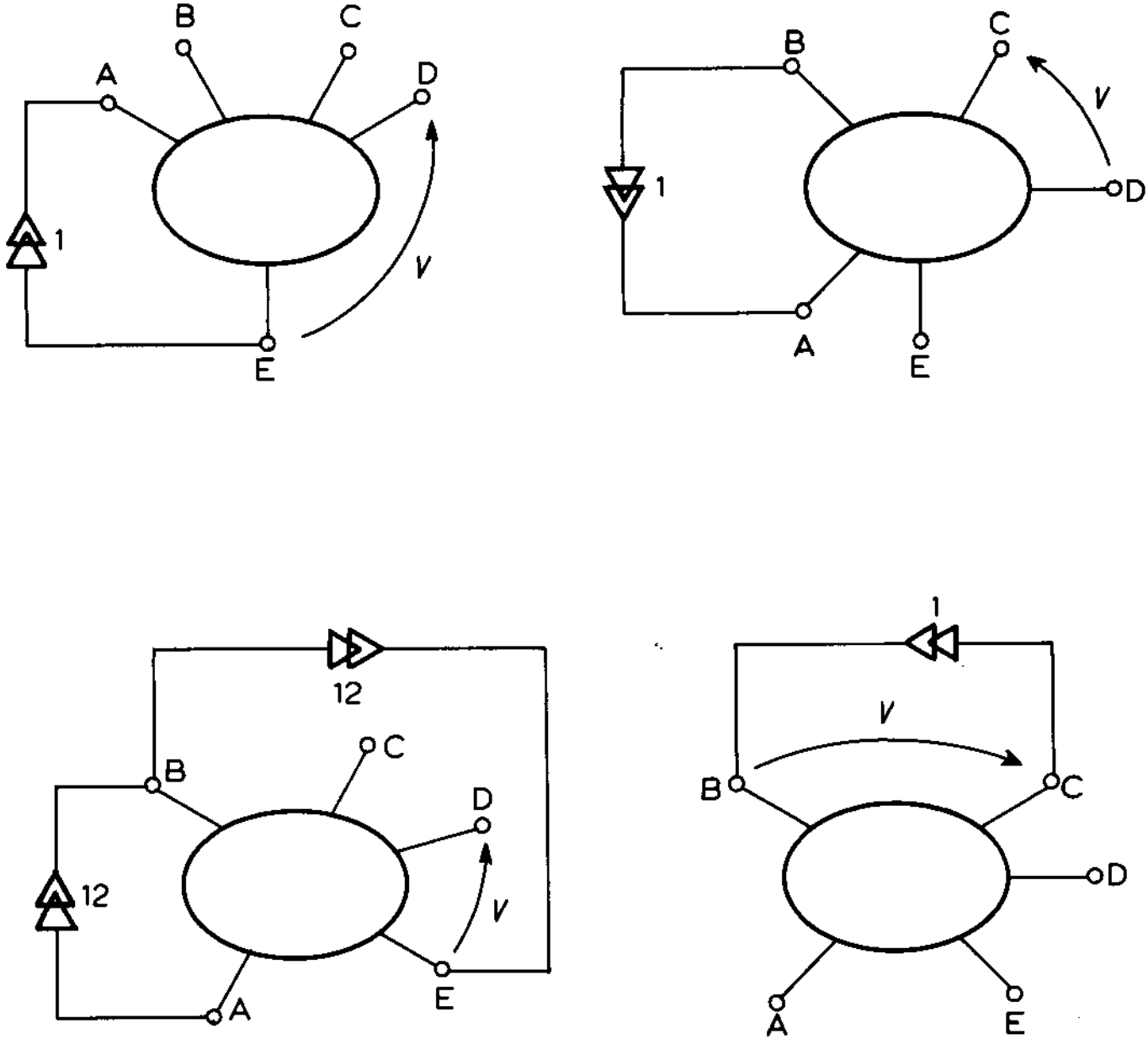
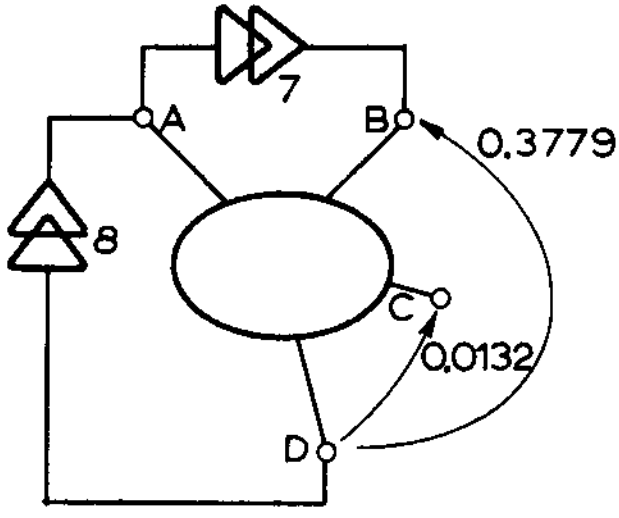
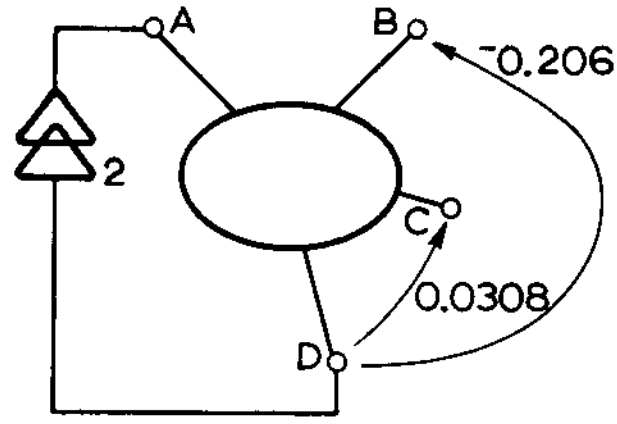


Fig. E.10·2



(a)



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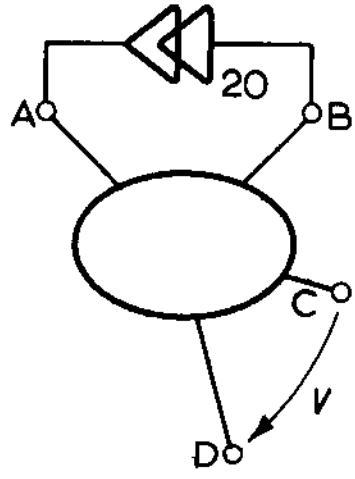
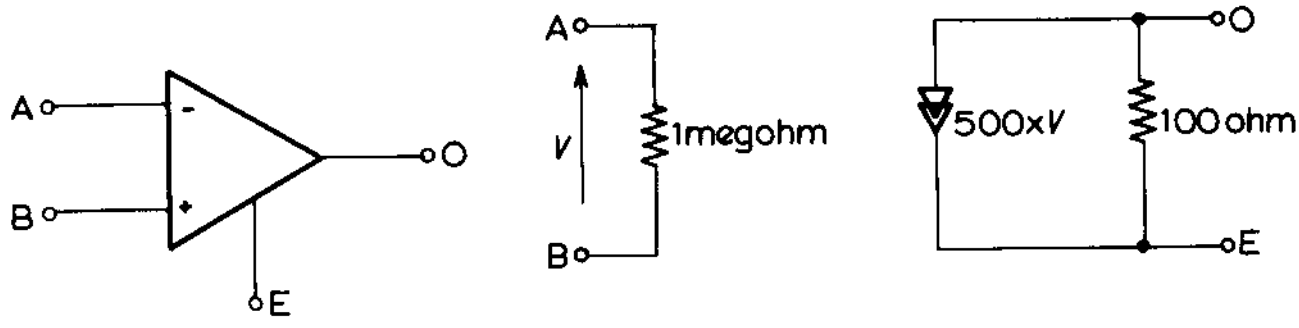
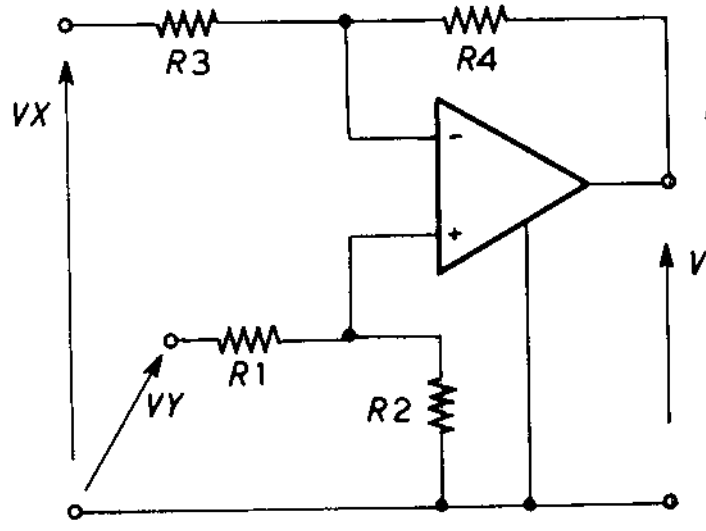


(c)

Fig. E.10.4

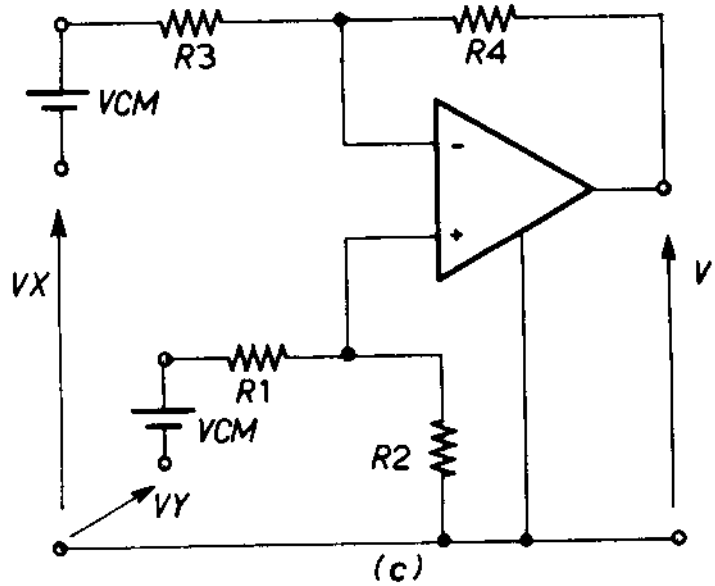


(a)



R1, R2, R3, and R4 are resistance values.

(b)



(c)

Fig. E. 10.5

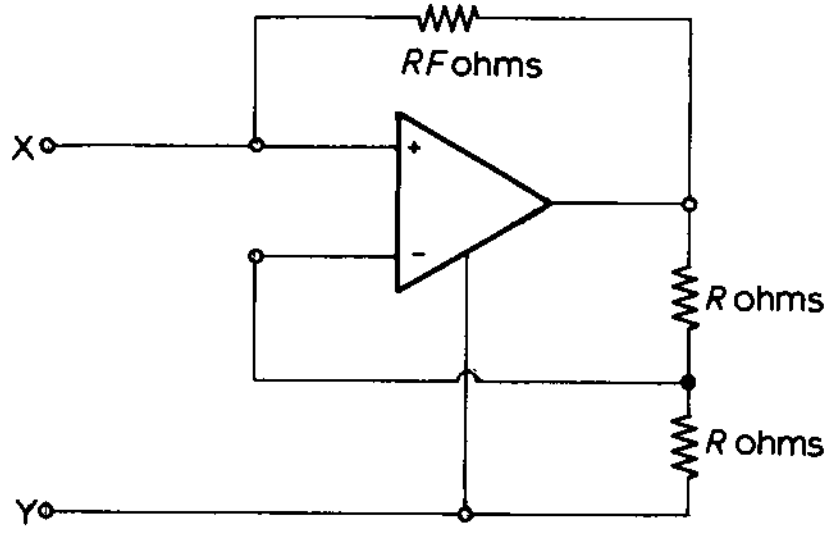


Fig. E. 10. 6.

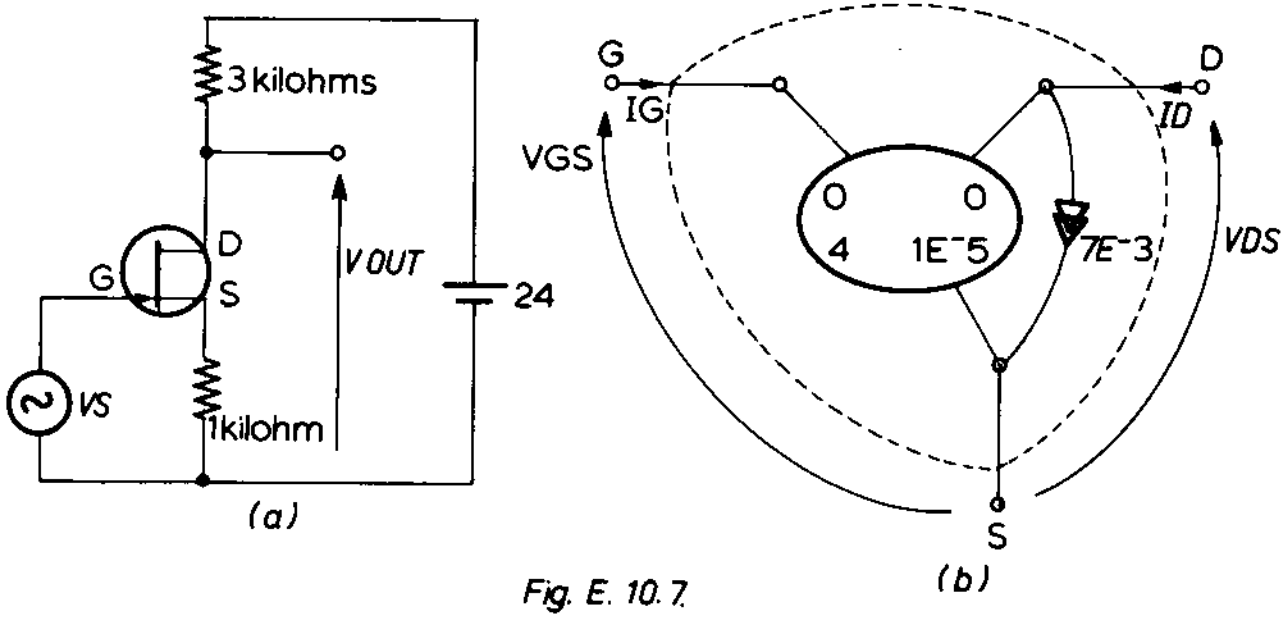


Fig. E. 10. 7.

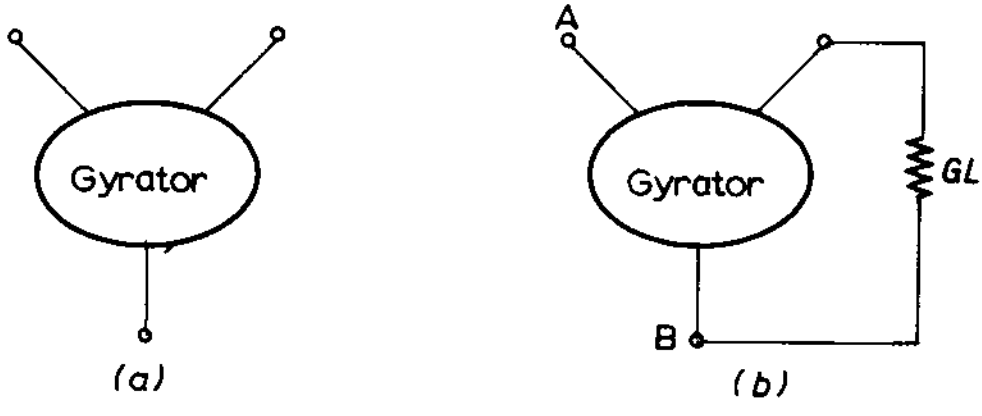


Fig. E. 10. 8.

EXERCISES: CHAPTER 11

1. A 2-terminal component whose current-voltage relation is simulated by the function *NONV*

```
      ∇I←NONV V
[1]   I←(V*3)+.01×V*5  ∇
```

is connected to a current source of 30 amps, as shown in Fig. E.11.1. By suitably modifving the function *ITER*, determine the component's voltage response *V* to this current excitation. For an initial guess at the response voltage not equal to the actual value, obtain a plot of both *ERROR* and ⍟*ERROR* versus the iteration step.

2. Define a function which returns the current response *I* of a linear and homogeneous 2-terminal component subjected to a voltage excitation *V*, and a second function which generates the slope of *I* versus *V* for a given value of *V*. Then modifv the function *ITER* to obtain the response voltage *V* when the component is excited by a current *IEX*, and test it. How many iteration cycles are performed ?

3. For the circuit shown in Fig. E.11.2 determine the current flowing in the nonlinear component by replacing the remainder of the circuit by its Thevenin equivalent, and employing the load-line method of analysis. Repeat the analysis by means of the function *ITER*.

(3) 4. For the nonlinear circuit of Fig. E.11.2 test to see if energy is conserved.

5. Modify the function *ITER* to determine the voltage across the component within the dashed lines in Fig. E.11.3. The left-hand argument of the new function will now be a 2-element vector whose first and second elements are, respectively, the values of *IS* and *VS*.

(3) 6. For the nonlinear circuit of Fig.E.11.2, test to see if the substitution theorem is applicable.

7. For the components shown in Fig. E.11.4 write down the branch array *T*. Devise and define an APL function *COMP* which takes *T* as its left argument, the vector *VB* of branch voltages as its right argument, and returns the

vector *IB* of branch currents. Check the function.

8. The locked function *CIRCUIT* takes the incidence matrix and the branch array as its left and right arguments, respectively, and returns a single array, called *N* in the text, which describes the circuit.To obtain *A* and *T* from *N* , two functions (also locked) have been defined for your use:

```
A←FA N          T←FT N
```

Confirm that the functions behave as stated.

(7,8) 9. By reference to your answer to exercise 7, define a function *NET* whose left argument is *N* (see exercise 8), whose right argument is the reduced vector *VR* of nodal voltages, and which returns the reduced vector *IR* of nodal currents. Test the function for a simple circuit.

(7,8,9)

10. Determine the voltage *V* in the circuit of Fig. E.11.5 for the current excitation shown, by using the function *ITERNET* as described in the text and the function *NET* you have defined yourself. The locked function *DIFNET* is provided. Experiment with different initial guesses at the voltage response, and also with different error criteria (1 microamp is used in the text) and see how each affects the number of iterations that elapse before the error criterion is satisfied.

11. Add a voltage-controlled current source to the circuit of Fig. E.11.5 and repeat the calculation of *V*. Then calculate the branch voltages and currents, and check that Kirchhoff's laws are obeyed.

12. Define an APL function which performs the function ascribed to *DIFNET* in the text. Test your function against the locked function *DIFNET*provided.

13. The sources of excitation in a nonlinear circuit are sometimes more conveniently represented by ideal voltage sources in series with components. In view of this, and with reference to Fig. E.11.6, modify the function *ITERNET* so that its left argument is a matrix *EX* whose first and second rows are, respectively, the vectors of branch voltage sources *VBS* and branch current sources *IBS*, these being the only sources associated with the circuit. The functions *NET* and *DIFNET* will also require modification. Test the new function by means of the example shown in Fig. E.11.7.

14. We have seen in earlier chapters that components such as semiconductor diodes and transistors are characterised by exponential functions. Define functions similar in nature to *ITER*, *NET* and *DIFNET*, but capable of handling branches of the general form shown in Fig. E.11.8(a), and which permit the iterative analysis of circuits composed of such branches. Test your functions by computing the nodal voltages associated with the two circuits shown in Fig. E.11.8(b), where the diode and

transistor are respectively described by the functions *DIODEV* and *EBMOLL*:

```
      ∇I←DIODEV V
[1]   I←1E¯11×¯1+*40×V ∇

      ∇I←EBMOLL V;M
[1]   M←2 2 ⍴ 1E¯6×(¯40 38 38 ¯50)
[2]   I←M+.×¯1+*-V×40 ∇
```

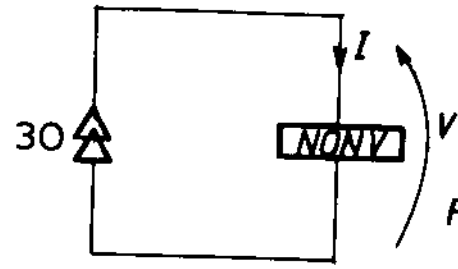


Fig. E. 11. 1.

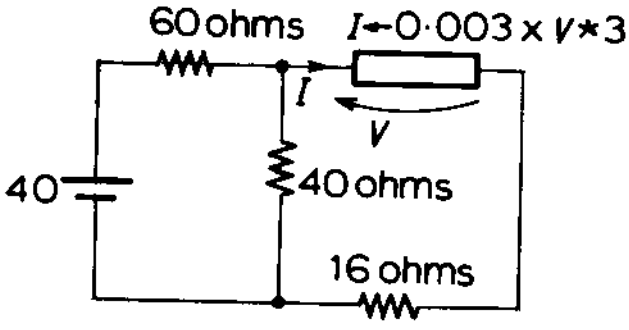


Fig. E. 11. 2

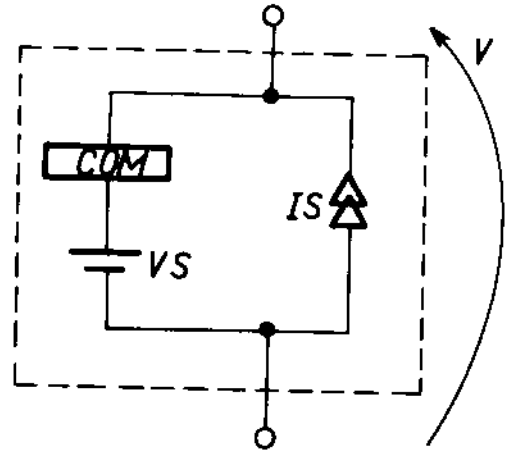
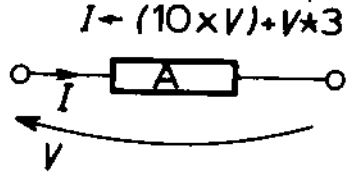


Fig. E. 11. 3



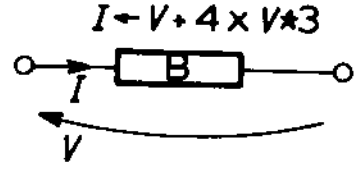
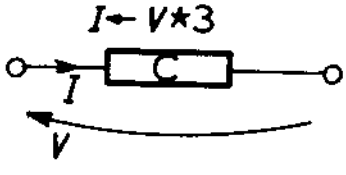
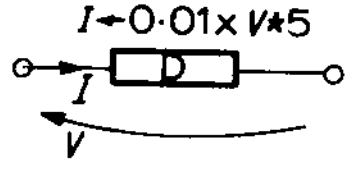


Fig. E. 11. 4.





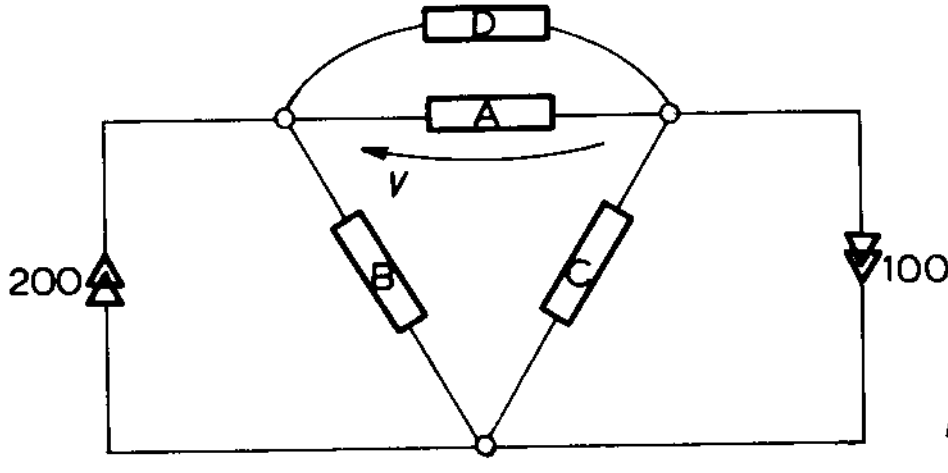


Fig. E. 11.5.

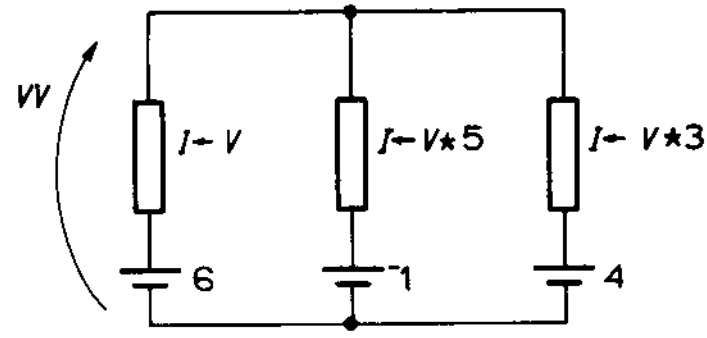
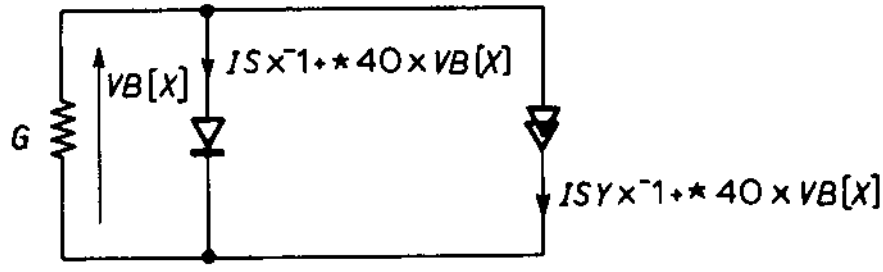
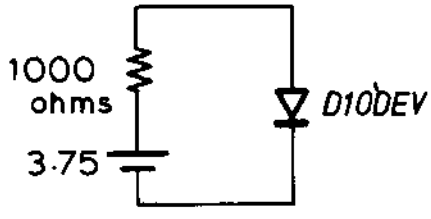
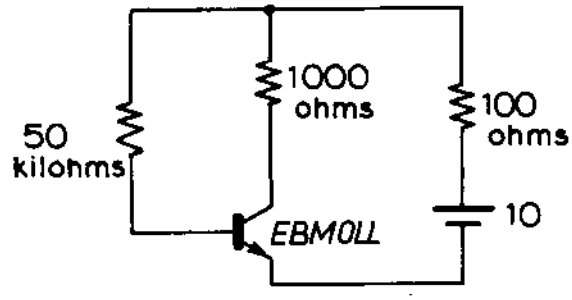


Fig. E. 11.7



(a)





(b)

Fig. E. 11·8

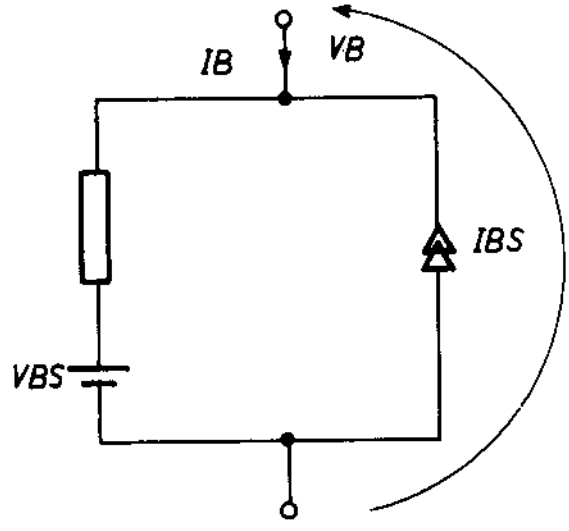


Fig. E. 11·6

EXERCISES: CHAPTER 12

1. A semiconductor diode is described by the function

```
      ∇I←DIODEV V
[1]   I←1E¯11×¯1+*40×V ∇
```

   Determine the local conductance *GD* of this diode for a large number of values of the quiescent voltage *VQ*. For each value of *VQ* compute the corresponding quiescent current *IQ*, and see if there is any simple relation between *GD* and either *VQ* or *IQ*. If you think you have found such a relation, obtain confirmation by attempting to derive it from the expression embodied in the function *DIODEV*.

2. Fig. E.12.1 shows a number of circuits containing both constant and timevarying sources of excitation. All signal sources are of 1 volt or 1 ampere peak-to-peak amplitude. For each circuit draw the small-signal equivalent circuit, and say if any constitute forbidden interconnections. Where a circuit is not forbidden, determine the peak-to-peak amplitudes of the signal voltages and currents identified respectively by *VS* and *IS*.

3. The circuit of Fig. E.12.2 is a simple variable gain circuit. Variation of the resistance *R* or the constant voltage source *E* changes the quiescent condition of the semiconductor diode (described by the function *DIODEV* see exercise 1) and hence the local conductance of the diode, thereby changing the small-signal voltage gain *VSO*÷*VSI*.First choose a number of values of *E* and determine the corresponding values of *IQ*, the quiescent current in the diode, and hence *GD*. Draw the small-signal equivalent of the circuit of Fig. E.12.2 and compute the small-signal voltage gain *VSO*÷*VSI* (it will be less than unity). Repeat for the situation in which the value of *R* varies. In each case obtain a plot of voltage gain versus the parameter (*E* or *R*) that is being varied.

4. Determine the quiescent condition of the circuit of Fig. E.12.3(a) by means of a load-line calculation. Now consider the application of a signal current excitation to the circuit, as shown in Fig. E.12.3(b). What is the value of the signal transfer resistance (i.e., the ratio of the signal voltage *VS* and the signal excitation current *IS*) provided the signal amplitudes are small ?

(9.18)

5. Is the substitution theorem valid for the small-signal components of a circuit's voltages and currents ? Test your answer by means of an example.

6. Fig. E.12.4 shows the measured characteristics of a (rather poor) transistor. Calculated the small-signal conductance matrix associated with the quiescent condition *VBE*←0.35, *VCE*←1. If, for this quiescent condition, the value of *G*[1;2] were to be about 1*E*¯5 (a more typical value for a transistor) how would the plot of *IB* versus *VBE* appear ?

7. The current-voltage relation describing a 3-terminal component is embodied in the function *BOXV* which takes a voltage argument whose first dimension is 2. For the quiescent condition *VQ*←5 ¯4 determine a linear homogeneous model which relates small changes in current and voltage. Test your model to see over what range of *V*[1] and *V*[2] the predicted (absolute) values of current are accurate to within 20 per cent.

8. The current-voltage relation of a 3-terminal NPN transistor (see Fig.6.10, page 87) is simulated by the function *NPNV*, which takes as its argument a voltage array whose first dimension is 2. The reference terminal is the base, and the nodes are ordered clockwise. For a number of quiescent conditions determine the small-signal conductance matrix *G* of the transistor, and search for a simple relation between the value of *G*[2;1] and the quiescent value of *I*[2] (the collector current). Derive such a relation under the assumption that the transistor is described by the Ebers-Moll model embodied in the function *TRANSISTOR* on page 90.

9. Fig. E.12.5(a) shows the circuit of a long-tailed pair differential amplifier. The signal components of its input and output voltages are denoted by *VSI* and *VSO* respectively. For small-signal operation each transistor may be modelled by the circuit of Fig. E.12.5(b), where

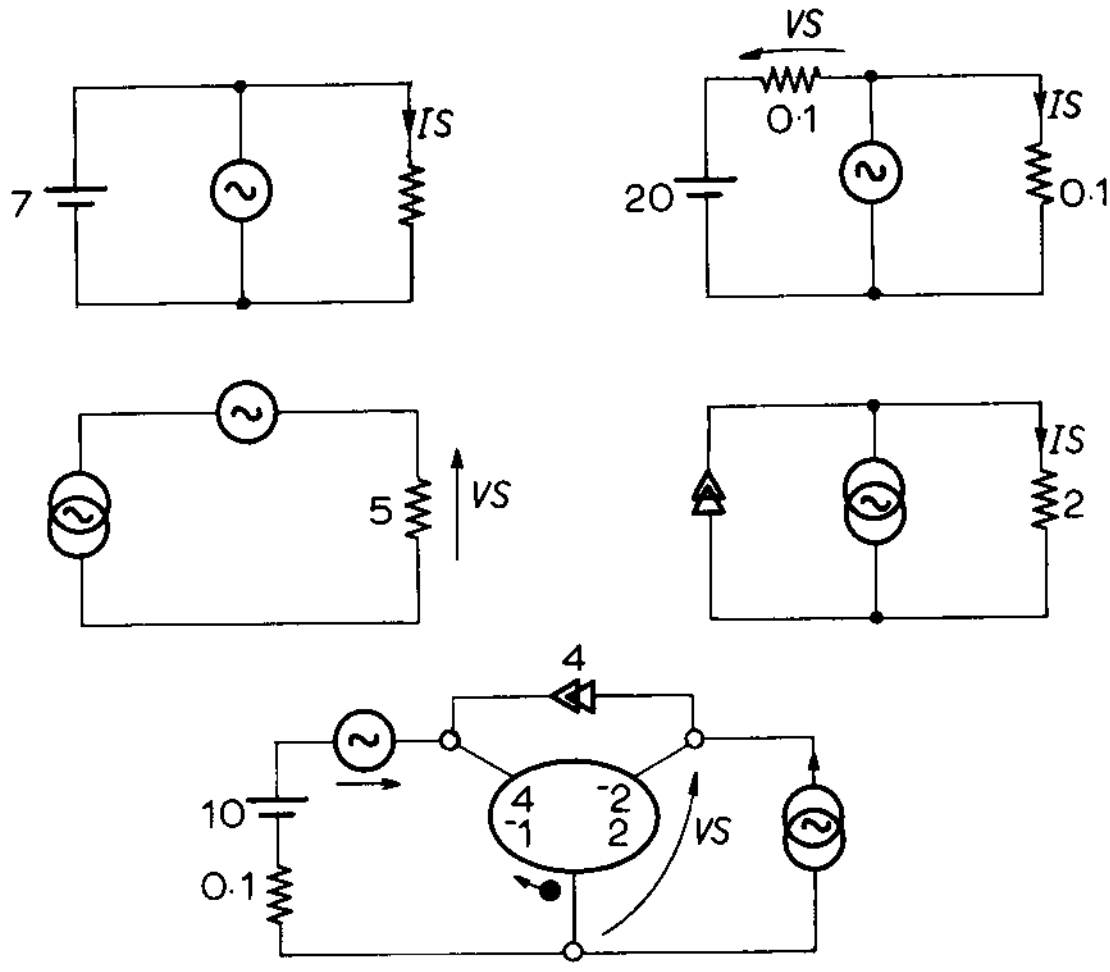
```
GM←40×ICQ
RPI←BETA÷GM
```

ICQ is the quiescent value of the collector current, and *BETA* is the base-to-collector current gain. If the quiescent collector currents have the values shown, what is the small-signal voltage amplification *VSO*÷*VSI* ?

10. The drain-source characteristics of a field-effect transistor (FET) are shown in Fig. E.12.6(a). It can be assumed that the gate current is zero. Part (b) of the figure shows a very simple amplifier circuit employing this transistor: the quiescent values of some currents and voltages are shown. Determine the quiescent condition of the FET and hence, from the characteristics, its small-signal conductance description. Compute the signal voltage amplification between the input and output terminal pairs.

(11.10)

11. A 1 ampere peak-to-peak amplitude source of signal excitation current is connected in parallel with components A and D in the circuit of Fig. E.11.5 (exercise 11.10). Use the reduced local nodal conductance matrix generated by the function *DIFNET* within *ITERNET* to determine the peak-to-peak amplitude of the voltage across component B. Check your answer by testing to see if Kirchhoff's laws are obeyed by the signal voltages and currents in the circuit.



Signal voltage and current sources have same waveform in reference directions indicated. 3-terminal component is linear and homogeneous.

Fig. E. 12. 1.

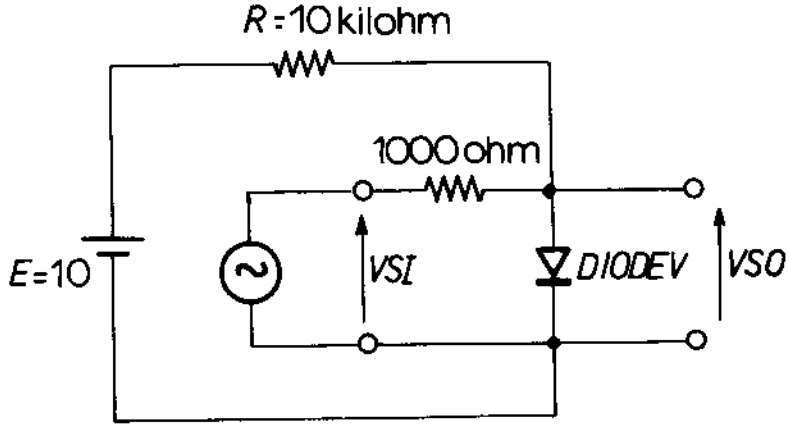


Fig. E. 12.2

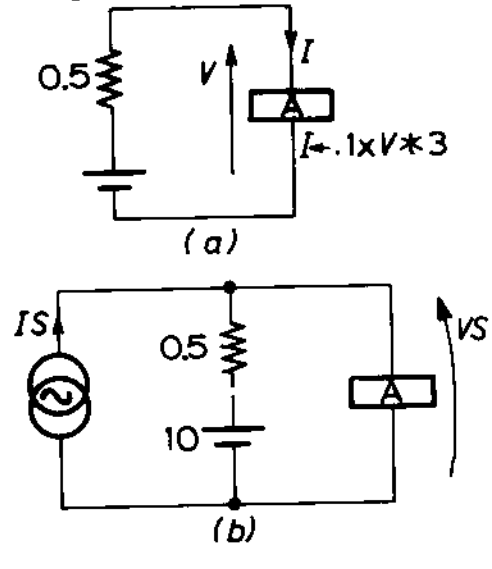


Fig. E. 12.3.

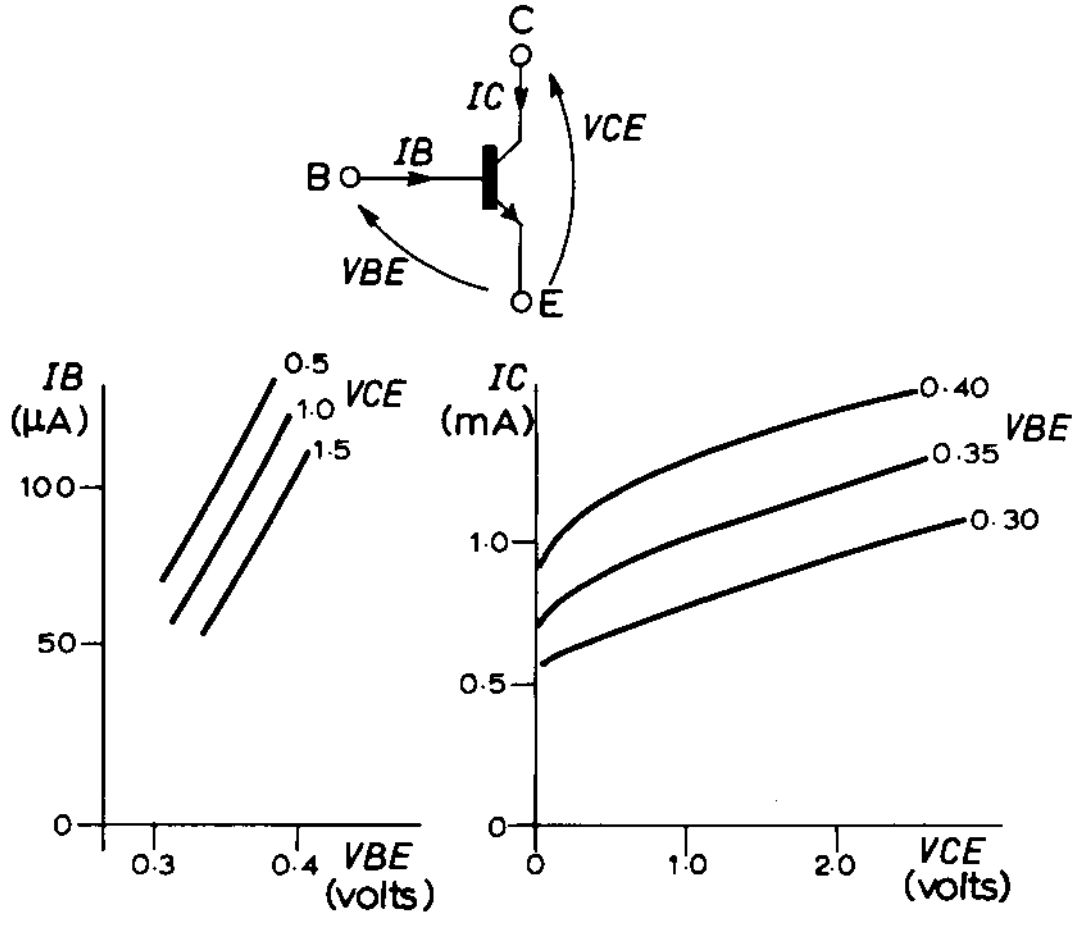
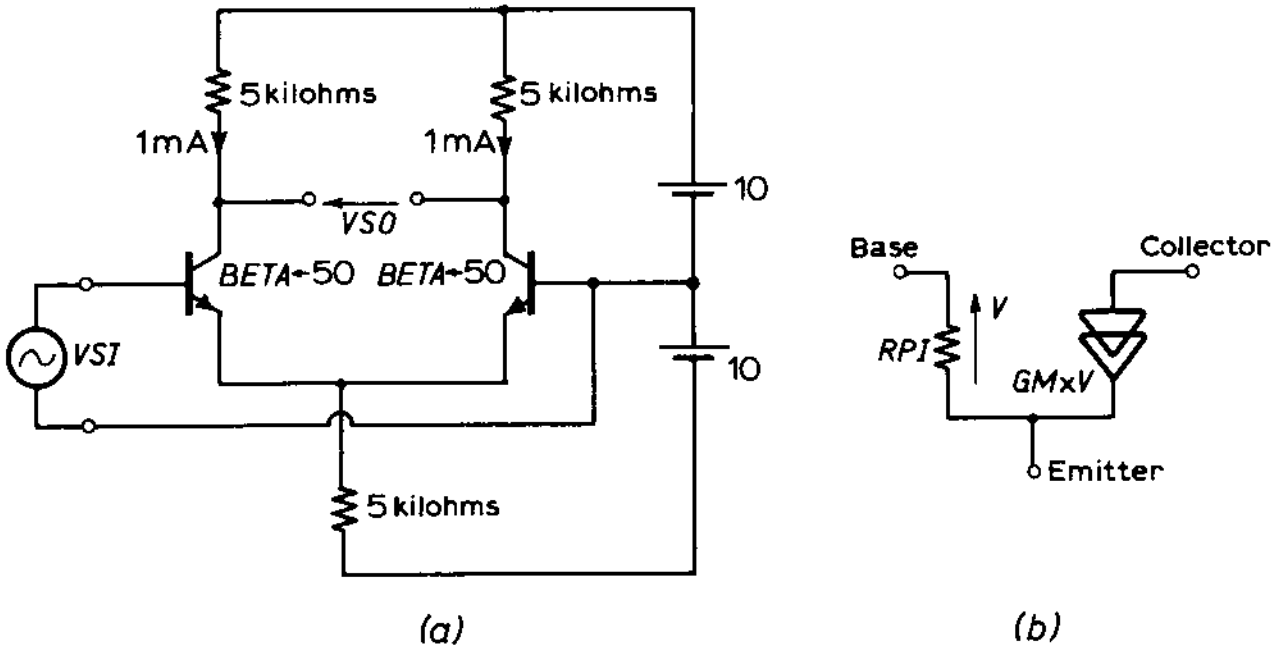
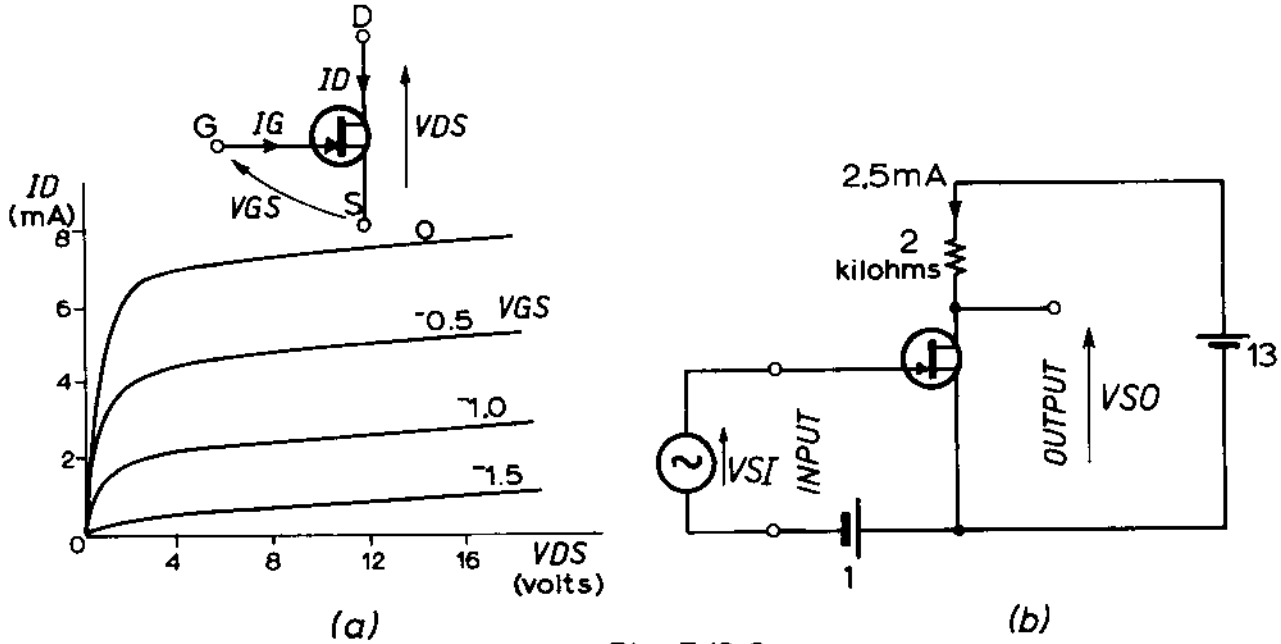

C
IC
VCE
IB
B
VBE
E
IB
(μA)
100
50
0
0.5
1.0
1.5
VCE
0.3
0.4
VBE
(volts)
IC
(mA)
1.0
0.5
0
0.40
0.35
0.30
VBE
0
1.0
2.0
VCE
(volts)

Fig. E. 12. 4



(a) (b)

Fig. E.12.5



(a) (b)

Fig. E.12.6

FUNCTION LISTINGS

Below are listed those APL functions which are referred to, but not defined, in the text and in the exercises.

```
BATTERYV1          ∇I←BATTERYV1 V
                [1] I←(V-5)÷10 ∇

BOXV               ∇I←BOX V;M;N
                [1] M←2 2 ⍴ 10 ¯7 ¯1 4
                [2] N←2 2 ⍴ 1E¯6
                [3] I←(M+.×V)+N+.×V*3 ∇

BOXV1              ∇I←BOXV1 V
                [1] I←(V-5)÷2 ∇

C1                 ∇I←C1 V
                [1] I←5×V ∇

C2                 ∇I←C2 V
                [1] I←(10×V)+.001×V*3 ∇

C3                 ∇I←C3 V
                [1] I←152.7×V ∇

CI1                ∇V←CI1 I
                [1] V←25.3×I ∇

CI2                ∇V←CI2 I
                [1] V←I*5 ∇

CI3                ∇V←CI3 I
                [1] V←1000×¯1+*2×I ∇

CI4                ∇V←CI4 I
                [1] V←I+(.1×I*3)+.01×I*5 ∇

CI5                ∇V←CI5 I
                [1] V←50×¯1+*5×I ∇

CIRCUIT            ∇N←A CIRCUIT T;B
                [1] N←T,[3](B,B,1)⍴,(B,B←1↑⍴T)↑A ∇

CIRCUITV           ∇I←CIRCUITV V
                [1] I←(6 6 ⍴ ¯20+⍳36)+.×V ∇

COM1               ∇I←COM1 V
                [1] I←V ∇
```

```
COM2                ∇I←COM2 V
                  [1] I←(V-5)×2 ∇

COMI1               ∇V←COMI1 I
                  [1] V←(I-1)×10 ∇

COMP                ∇IB←T COMP VB
                  [1] IB←+/+/T×(ρT)ρVB∘.*⍳¯1↑ρT ∇

COMV1               ∇I←COMV1 V
                  [1] I←127×V ∇

COMV2               ∇I←COMV2 V
                  [1] I←78×¯1+*376×V ∇

COMV3               ∇I←COMV3 V
                  [1] I←1E¯6×¯1+*20×V-6 ∇

COMV4               ∇I←COMV4 V
                  [1] I←.01+V*5 ∇

COMV5               ∇I←COMV5 V
                  [1] I←(2 2 ρ 10 2 ¯10 20)+.×V ∇

COMV6               ∇I←COMV6 V
                  [1] I←(2 2 ρ 4 ¯8 ¯1 20)+.×V ∇

COMV7               ∇I←COMV7 V
                  [1] →2×∧/2=ρV
                  [2] I←(2 2 ρ 4 3 2 1)+.×V ∇

COMV9               ∇I←COMV9 V
                  [1] →2×∧/2=ρV
                  [2] I←(2 2 ρ 20 ¯2 ¯8 40)+.×V ∇

COMV10              ∇I←COMV10 V
                  [1] I←(2 2 ρ 0 1 2 1)+.×V ∇

COMV11              ∇I←COMV11 V
                  [1] I←(2 2 ρ 2 0 2 1)+.×V ∇

COMV12              ∇I←COMV12 V
                  [1] I←(2 2 ρ 1 2 0 1)+.×V ∇

COMV13              ∇I←COMV13 V
                  [1] I←(2 2 ρ 1 2 1 0)+.×V ∇

COMV14              ∇I←COMV14 V
                  [1] I←(2 2 ρ 1 10 ¯5 3)+.×V ∇

CR1                 ∇V←CR1 R
                  [1] V←R×100÷50+R ∇

CR2                 ∇I←CR2 R
                  [1] I←¯27÷10+R ∇
```

```
CR3                 ∇I←CR3 R
                  [1] I←30÷100+R ∇

CV1                 ∇I←CV1 V
                  [1] I←(5×V)+(.12×V*2)+.01×V*3 ∇

CV2                 ∇I←CV2 V
                  [1] I←V*3 ∇

CV3                 ∇I←CV3 V
                  [1] I←(100×V)+V*3 ∇

CV4                 ∇I←CV4 V
                  [1] I←(10×V)+.5×V*2 ∇

CV5                 ∇I←CV5 V
                  [1] I←(50×V)+¯1+*.25×V ∇

CV6                 ∇I←CV6 V
                  [1] I←(100×V)+¯1+*V ∇

CV7                 ∇I←CV7 V
                  [1] I←22×V ∇

CV8                 ∇I←CV8 V
                  [1] I←(V-8)÷2 ∇

CV9                 ∇I←CV9 V
                  [1] I←(10×V)+¯1+*V ∇

CV10                ∇I←CV10 V
                  [1] I←0.01×¯1+*4×V ∇

CV11                ∇I←CV11 V
                  [1] I←(100×V)+¯1+*2×V ∇

CV12                ∇I←CV12 V
                  [1] I←V+(V*3)+.002×V*5 ∇

DIFNET              ∇S←N DIFNET VR;A;T;VB;HP;YBI
                  [1] A←FA N
                  [2] T←FT N
                  [3] VB←(⌽A)+.×VR,0
                  [4] YBI←+/T×((ρT)ρ⍳HP)×(ρT)ρVB∘.*¯1+⍳HP←¯1↑ρT
                  [5] S←¯1 ¯1 ↓ A+.×YBI+.×⌽A ∇

DIODEV              ∇I←DIODEV V
                  [1] I←1E¯11×¯1+*40×V ∇

DV                  ∇I←DV V
                  [1] I←1E¯10×¯1+*37×V ∇
```

```
EBMOLL          ∇I←EBMOLL V;M
                [1] M←2 2 ρ 1E¯6×(¯40 38 38 ¯50)
                [2] I←M+.×¯1+⍟-V×40 ∇

F1              ∇I←F1 V
                [1] I←(2×V)+.2×V*2 ∇

FA              ∇A←FA N;X;S
                [1] S←+/,∨/[1]|(X,(X←1↑ρN),¯1)↑N
                [2] A←(S,X)ρ, 3 2 1 ⍉N[;⍳S;¯1↑ρN] ∇

FFV1            ∇I←FFV1 V
                [1] I←¯10+2×V ∇

FFV2            ∇I←FFV2 V
                [1] I←.1×V*3 ∇

FI              ∇V←FI I
                [1] V←I⍎2 ∇

FT              ∇T←FT N
                [1] T←0 0 ¯1↓N ∇

FV              ∇I←FV V
                [1] I←2×V ∇

GPP             ∇GP←G1 GPP G2
                [1] GP←G1+G2 ∇

INVC1           ∇V←INVC1 I
                [1] V←.2×I ∇

NET             ∇IR←N NET VR;A;T;VB;IB
                [1] A←FA N
                [2] T←FT N
                [3] VB←(⍉A)+.×VR,0
                [4] IB←T COMP VB
                [5] IR←¯1↓A+.×IB ∇

NONV            ∇I←NONV V
                [1] I←(V*3)+.01×V*5 ∇

NPNV            ∇I←NPNV V;M
                [1] M←2 2 ρ 1E¯6×(¯50 47 47 ¯60)
                [2] I←M+.×¯1+⍟-40×V ∇

ONEBOXI1        ∇V←ONEBOXI1 I
                [1] V←2×I ∇

ONEV4           ∇I←ONEV4 V
                [1] I←(4×V)+.1×V*3 ∇
```

```
ONEV20          ∇I←ONEV20 V
              [1] I←(10×V)+.5×V*3 ∇

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PNPV            ∇I←PNPV V;M;N
              [1] M←2 2 ρ 1E¯6×(60 ¯58 ¯58 70)
              [2] N←2 2 ρ 1E¯5×(1 ¯1 ¯1 1)
              [3] I←(N+.×V)+M+.×¯1+*42×V ∇

PSUP            ∇P←PSUP R
              [1] P←R×(100÷10+R)*2 ∇

SEMIV1          ∇I←SEMIV1 V
              [1] I←4E¯11×¯1+*41×V ∇

TRANS           ∇I←TRANS V;M
              [1] M←2 2 ρ 1E¯6×(¯40 38 38 ¯50)
              [2] I←M+.×¯1+*-40×V ∇

TWOV1           ∇I←TWOV1 V
              [1] I←(2 2 ρ 3 ¯2 ¯1 4)+.×V ∇

TWOV6           ∇I←TWOV6 V;M;N
              [1] M←2 2 ρ 2 3 1 0.4
              [2] N←2 2 ρ 0.2 0.1 0.5 0.05
              [3] I←(M+.×V)+N+.×V*3 ∇

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WITH            NORMALLY AVAILABLE FROM THE
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